The Spectral Radius Theorem

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In this document, we give a proof of the famous *Spectral Radius Theorem* for unital Banach algebras over $\mathbb{C}$. The proof we give is that covered in a topics course (Math 595) given at McGill University during the winter semester of 2018. In turn, this proof follows closely that presented in the Bratteli-Robinson monograph *Operator Algebras and Quantum Statistical Mechanics*.

Throughout this document, unless stated otherwise, $\mathfrak{A}$ denotes a unital Banach algebra over the complex numbers and $\mathbb{1}$ denotes its multiplicative identity. The result we prove is the following:

Theorem. *Let $\mathfrak{A}$ be a Banach algebra with identity and fix $A \in \mathfrak{A}$. We define the spectral radius of A as*

$$\rho(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Then, $\sigma(A)$ is non-empty, compact and

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} \leq \|A\|. \tag{1}$$

The proof of this theorem is slightly involved and we will first require some preliminary lemmas. This is what the first section accomplishes.

1 Very Useful Lemmas

All Banach algebras are assumed to have an identity $\mathbb{1}$.

Lemma 1. *Let $\mathfrak{A}$ be a Banach algebra and let $A \in \mathfrak{A}$. Suppose that*

$$\sum_{n=0}^{\infty} \left\| \frac{A^n}{\lambda^{n+1}} \right\| = \sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} < \infty$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\lambda \in r(A)$, i.e.

$$(\lambda\mathbb{1} - A) \text{ is invertible.}$$

Moreover, the inverse of $(\lambda\mathbb{1} - A)$ is given by

$$\sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}.$$

In particular, this holds whenever $|\lambda| > \|A\|$.

Proof of Lemma 1. By hypothesis,

$$\sum_{n=0}^{\infty} \left\| \frac{A^n}{\lambda^{n+1}} \right\| = \sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} < \infty.$$

Since a normed complex vector space V is complete if and only if every absolutely convergent series converges in V, the series given by

$$\sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{A^n}{\lambda^{n+1}} =: \lim_{N\to\infty} S_N$$

exists in the Banach algebra $\mathfrak{A}$. Let $S \in \mathfrak{A}$ denote the limit of the sequence $(S_N)_{N\geq 0}$. We claim that S is the inverse of $(\lambda\mathbb{1} - A)$. To verify this, we will show that

$$(\lambda\mathbb{1} - A)S = \mathbb{1},$$

as a symmetric argument yields $S(\lambda\mathbb{1} - A) = \mathbb{1}$. Now, since $\mathfrak{A}$ is a Banach algebra, one has for every $N \geq 0$

$$\|(\lambda\mathbb{1} - A)S_N - (\lambda\mathbb{1} - A)S\| = \|(\lambda\mathbb{1} - A)(S_N - S)\| \leq \|\lambda\mathbb{1} - A\| \, \|S_N - S\| \xrightarrow{N\to\infty} 0$$

since $S_N \to S$ in $\mathfrak{A}$. This is precisely the statement that

$$\lim_{N\to\infty} (\lambda\mathbb{1} - A)S_N = (\lambda\mathbb{1} - A)S.$$

On the other hand, direct computation gives for all $N \geq 2$

$$(\lambda\mathbb{1} - A)S_N = \sum_{n=0}^{N} \frac{\lambda A^n}{\lambda^{n+1}} - A\sum_{n=0}^{N} \frac{A^n}{\lambda^{n+1}} = \sum_{n=0}^{N} \frac{A^n}{\lambda^n} - \sum_{n=1}^{N+1} \frac{A^n}{\lambda^n} = \mathbb{1} - \frac{A^{N+1}}{\lambda^{N+1}}.$$

Since the series $\sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} = \frac{1}{|\lambda|} \sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^n}$ is convergent in $\mathbb{R}$, we must have

$$\left\| \frac{A^{N+1}}{\lambda^{N+1}} \right\| = \frac{\left\| A^{N+1} \right\|}{|\lambda^{N+1}|} \xrightarrow{N\to\infty} 0.$$

But, it then follows that

$$\lim_{N\to\infty}(\lambda\mathbb{1}-A)S_N=\lim_{N\to\infty}\left[\mathbb{1}-\frac{A^{N+1}}{\lambda^{N+1}}\right]=\mathbb{1}.$$

By uniqueness of the limit, $\mathbb{1}=(\lambda\mathbb{1}-A)S$. As mentioned above, a similar argument shows that $\mathbb{1}=S(\lambda\mathbb{1}-A)$ and thus we have that

$$(\lambda\mathbb{1}-A)^{-1}=S.$$

In particular, $\lambda\in r(A)$. □

Lemma 2. *Let $\mathfrak{A}$ be a Banach algebra and suppose that $(A_n)_{n\geq 0}$ is a sequence in $\mathfrak{A}$ such that the series*

$$S:=\sum_{n=0}^{\infty}A_n$$

converges absolutely in $\mathfrak{A}$. Let $\pi:\mathbb{N}_0\to\mathbb{N}_0$ be a bijection and consider the sequence $(B_n)_{n\geq 0}$ defined by $B_n:=A_{\pi(n)}$. Then

$$\sum_{n=0}^{\infty}B_n=S.$$

Also, the series $\sum_{n=0}^{\infty}B_n$ converges absolutely in $\mathfrak{A}$.

Proof of Lemma 2. Given that π is a bijection, it is clear that

$$\sum_{n=0}^{\infty}\|A_n\|=\sum_{n=0}^{\infty}\|B_n\|$$

and so the latter series also converges absolutely. A complex normed space is complete if and only if every absolutely convergent series has a limit in the space. Since $\mathfrak{A}$ is by definition a Banach space over $\mathbb{C}$, the element $S\in\mathfrak{A}$ is well defined. For given $N,M\in\mathbb{N}$, let S_N and T_M denote the partial sums:

$$S_N:=\sum_{n=0}^{N}A_n,\quad T_M:=\sum_{m=0}^{M}B_m.$$

Let $\varepsilon>0$ be given, there exists $K\in\mathbb{N}$ so large that

$$\|S_N-S\|<\varepsilon\quad\text{and}\quad\sum_{n\geq N+1}\|A_n\|<\varepsilon.$$

whenever $N \geq K$. Indeed, this follows from the fact that $S_N \to S$ in $\mathfrak{A}$ and $\sum_{n=0}^{\infty} \|A_n\| < \infty$. Choose $K' \in \mathbb{N}$ such that

$$\{A_0, A_1, \ldots, A_K\} \subseteq \left\{A_{\pi(0)}, A_{\pi(1)}, \ldots, A_{\pi(K')}\right\} = \{B_0, B_1, \ldots, B_{K'}\}.$$

Let now $M \geq K'$ and notice that $T_M - S_K$ is a finite number of A_n's with $n > K$. This implies that

$$\|T_M - S_K\| \leq \sum_{n>K} \|A_n\| < \varepsilon.$$

It easily follows from this that for all $M \geq K'$:

$$\|T_M - S\| \leq \|T_M - S_K\| + \|S_K - S\| < 2\varepsilon.$$

Since ε was arbitrary, we see that

$$S = \lim_{m\to\infty} T_m = \sum_{m=0}^{\infty} B_m.$$

□

2 The Proof of the Spectral Radius Theorem

We will decompose the proof into multiple steps which, when put together, will yield the theorem.

Step 2.1. *Let $\lambda \in \mathbb{C}$ and suppose that $|\lambda|^m > \|A^m\|$ for some $m \in \mathbb{N}$. Then $\lambda \in r(A)$. Moreover,*

$$(\lambda \mathbb{1} - A)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}$$

where the series on the right hand side converges absolutely in $\mathfrak{A}$. In particular, one has

$$\lim_{n\to\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} = \lim_{n\to\infty} \frac{\|A^n\|}{|\lambda|^n} = 0$$

by absolute convergence.

Proof of Step 2.1. Suppose for the moment that we can show that

$$\sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} = \sum_{n=0}^{\infty} \left\| \frac{A^n}{\lambda^{n+1}} \right\| < \infty. \tag{2}$$

Then, we would be done by Lemma 1.

Thus, we are now reduced to proving (2). Let us first consider of the case $m = 1$. Clearly, $|\lambda| > \|A\|$ so that

$$\sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} \le \frac{1}{|\lambda|} \sum_{n=0}^{\infty} \left[\frac{\|A\|}{|\lambda|} \right]^n < \infty$$

since this is a well known geometric series. Therefore, we may assume that $m \ge 2$. For each $n \in \mathbb{N}$, we may *uniquely* choose $p, q \in \mathbb{N}_0$ with $q < m$ such that

$$n = pm + q.$$

This yields

$$\frac{\|A^n\|}{|\lambda|^{n+1}} = \frac{\|A^{pm+q}\|}{|\lambda|^{pm+q+1}} \le \left(\frac{\|A^m\|}{|\lambda|^m} \right)^p \frac{\|A\|^q}{|\lambda|^{q+1}}.$$

Therefore, a natural series of estimates gives

$$\sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^{n+1}} \le \sum_{p=0}^{\infty} \sum_{q=0}^{m-1} \left(\frac{\|A^m\|}{|\lambda|^m} \right)^p \frac{\|A\|^q}{|\lambda|^{q+1}} = \left[\sum_{q=0}^{m-1} \frac{\|A\|^q}{|\lambda|^{q+1}} \right] \sum_{p=0}^{\infty} \left(\frac{\|A^m\|}{|\lambda|^m} \right)^p$$

where this last term is finite by well known properties of a geometric series. This shows that (2) holds and the proof of this step is complete. □

Step 2.2. *The spectrum* $\sigma(A)$ *is non-empty. Also,*

$$\rho(A) \le \inf_{n \in \mathbb{N}} \|A^n\|^{1/n} \le \liminf_{n \to \infty} \|A^n\|^{1/n}. \tag{3}$$

Proof of Step 2.2. For the moment, assume that $\sigma(A) \neq \varnothing$. By Step 2.1, if $\lambda \in \mathbb{C}$ satisfies the inequality $|\lambda| > \|A^m\|^{1/m}$ for some $m \in \mathbb{N}$, then $\lambda \notin \sigma(A)$. This implies that $\rho(A) \le \|A^m\|^{1/m}$ for all $m \in \mathbb{N}$ and therefore

$$\rho(A) \le \inf_{m \in \mathbb{N}} \|A^m\|^{1/m} \le \liminf_{n \to \infty} \|A^n\|^{1/n}. \tag{4}$$

We now relax the assumption that $\sigma(A) \neq \varnothing$. Define

$$r_A := \limsup_{n \to \infty} \|A^n\|^{1/n} \le \|A\| < \infty.$$

Clearly, $\rho(A) \leq r_A$. We now distinguish two possible cases.

Case 1: *There holds* $r_A = 0$. We claim that A cannot be invertible. To see this, we argue by contradiction. If A is invertible, then for every $n \in \mathbb{N}$ the element A^n would be invertible. Thus,

$$1 = \|\mathbb{1}\| = \|A^n A^{-n}\| \leq \|A^n\| \, \|A^{-n}\| \, .$$

This implies that $1 \leq \|A^n\|^{1/n} \, \|A^{-n}\|^{1/n}$ for all $n \in \mathbb{N}$. Now,

$$\|A^{-n}\|^{1/n} \leq \left\|A^{-1}\right\| < \infty.$$

Therefore,

$$r_{A^{-1}} = \limsup_{n\to\infty} \|A^{-n}\|^{1/n} < \infty.$$

This means that

$$1 \leq \limsup_{n\to\infty} \left(\|A^n\|^{1/n} \, \|A^{-n}\|^{1/n} \right) \leq r_A r_{A^{-1}} = 0$$

which is a contradiction. Hence, A cannot be invertible. This implies that $0 \in \sigma(A)$ and that the spectrum is non-empty. Notice that by the arguments at the start of this step, (3) holds true. In fact, using (4) it follows that

$$0 \leq \rho(A) \leq \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} \leq \liminf_{n\to\infty} \|A^n\|^{1/n} \leq r_A = 0$$

and so we must have equality throughout, i.e.

$$\rho(A) = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} = \liminf_{n\to\infty} \|A^n\|^{1/n} = r_A = 0.$$

Case 2. *One has* $r_A > 0$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that

$$R_n = (\mathbb{1} - A_n)^{-1} \quad \text{exists in } \mathfrak{A} \text{ for all } n.$$

By taking $A_n = 0$ for every n, we see such a sequence indeed exists. In any case, for each $n \in \mathbb{N}$, a direct calculation gives

$$\begin{aligned} \mathbb{1} - R_n = \mathbb{1} - (\mathbb{1} - A_n)^{-1} &= (\mathbb{1} - A_n)(\mathbb{1} - A_n)^{-1} - (\mathbb{1} - A_n)^{-1} \\ &= -A_n \, (\mathbb{1} - A_n)^{-1} \end{aligned}$$

and a similar calculation gives

$$\begin{aligned} -(\mathbb{1} - R_n)R_n^{-1} = -(\mathbb{1} - R_n)(\mathbb{1} - A_n) &= A_n + R_n - \mathbb{1} - R_n A_n \\ &= A_n + R_n(\mathbb{1} - A_n) - \mathbb{1} \\ &= A_n. \end{aligned}$$

From which we have

$$A_n = -(\mathbb{1} - R_n)R_n^{-1} = -(\mathbb{1} - R_n)\left(\mathbb{1} - (\mathbb{1} - R_n)\right)^{-1}.$$

We now claim that, as $n \to \infty$, $\|A_n\| \to 0$ if and only if $\|\mathbb{1} - R_n\| \to 0$. Suppose that $\|A_n\| \to 0$; then for all sufficiently large n one has $\|A_n\| < 1/2$. We now write

$$\begin{aligned}\|\mathbb{1} - R_n\| = \left\|A_n(\mathbb{1} - A_n)^{-1}\right\| \leq \|A_n\| \left\|(\mathbb{1} - A_n)^{-1}\right\| &= \|A_n\| \left\|\sum_{k=0}^{\infty} A_n^k\right\| \\ &\leq \|A_n\| \sum_{k=0}^{\infty} \left\|A_n^k\right\| \\ &\leq \frac{\|A_n\|}{1 - \|A_n\|} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Here, we have applied Lemma 1 to A_n with $\lambda = 1 > \|A_n\|$ in order to obtain a "power series expansion" for $(\mathbb{1} - A_n)^{-1}$ in $\mathfrak{A}$. Also, we have made use of the identity

$$\begin{aligned}\mathbb{1} - R_n = \mathbb{1} - (\mathbb{1} - A_n)^{-1} &= (\mathbb{1} - A_n)(\mathbb{1} - A_n)^{-1} - (\mathbb{1} - A_n)^{-1} \\ &= -A_n(\mathbb{1} - A_n)^{-1}\end{aligned}$$

in the first equality. Conversely, suppose that $\|\mathbb{1} - A_n\| \to 0$ as $n \to \infty$. Writing

$$A_n = \mathbb{1} - R_n^{-1} = -(\mathbb{1} - R_n)R_n^{-1} = -(\mathbb{1} - R_n)\left(\mathbb{1} - (\mathbb{1} - R_n)\right)^{-1}$$

we see that one may apply the same argument to $(\mathbb{1} - R_n)$ instead of A_n. Thus,

$$\lim_{n\to\infty} \|A_n\| = 0 \iff \lim_{n\to\infty} \|\mathbb{1} - R_n\| = 0 \tag{5}$$

By way of contradiction, we will assume that

$$S_A := \{\lambda \in \mathbb{C} : |\lambda| \geq r_A\} \subseteq r(A).$$

If the above inclusion does not hold, then clearly $\sigma(A) \neq \varnothing$ and $\rho(A) \geq r_A$. We take a quick "break" from the proof and instead develop the following lemma.

Lemma 3. *For $n \in \mathbb{N}$ let $\omega := e^{2\pi i/n}$ and define for $|\lambda| \geq r_A$.*

$$R_n(\lambda, A) := \frac{1}{n} \sum_{k=1}^{n} \left(\mathbb{1} - \frac{\omega^k}{\lambda} A\right)^{-1}.$$

Then

$$R_n(\lambda, A) = \left[\mathbb{1} - \frac{A^n}{\lambda^n}\right]^{-1}.$$

Proof. We first point out that the statement "makes sense". Indeed, by hypothesis, $S_A \subseteq r(A)$ and therefore

$$\left(\frac{\lambda}{\omega^k}\mathbb{1} - A\right) \text{ is invertible.}$$

Suppose first that $|\lambda| > r_A$. Then there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ one has $|\lambda|^k > \left\|A^k\right\|$. Let $n \in \mathbb{N}$ be given and define

$$\mu := \lambda^n \quad \text{and} \quad B := A^n.$$

Let $m \in \mathbb{N}$ be so large that $nm > N_0$. Then,

$$|\mu|^m = |\lambda|^{nm} > \|A^{nm}\| = \|B^m\|.$$

By Step 2.1, we see that $\mu\mathbb{1} - B$ is invertible with inverse

$$(\lambda^n\mathbb{1} - A^n)^{-1} = (\mu\mathbb{1} - B)^{-1} = \sum_{j=0}^{\infty} \frac{B^j}{\mu^{j+1}} = \sum_{j=0}^{\infty} \frac{(A^n)^j}{(\lambda^n)^{j+1}}$$

Thus, for any $n \in \mathbb{N}$ there holds

$$\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{A^n}{\lambda^n}\right)^j \tag{6}$$

Observe that $|\omega| = 1$. By the same argument, we may choose $m \in \mathbb{N}$ such that

$$\left(\frac{\lambda}{\omega^k}\right)^m > \|A^m\|$$

and thus by Step 2.1 we have

$$\left(\mathbb{1} - \frac{\omega^k A}{\lambda}\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{\omega^k A}{\lambda}\right)^j$$

where the series on the right converges absolutely in $\mathfrak{A}$. An easy calculation gives

$$\sum_{k=1}^{n} \left(\omega^k\right)^j = \begin{cases} n & \text{if } n \mid j \\ 0 & \text{otherwise} \end{cases}$$

Invoking our "rearrangement lemma" (Lemma 2), we have

$$\sum_{k=1}^{n}\left(\mathbb{1}-\frac{\omega^k A}{\lambda}\right)^{-1}=\sum_{k=1}^{n}\sum_{\ell=0}^{\infty}\left(\frac{\omega^k A}{\lambda}\right)^{\ell}=\sum_{\ell=0}^{\infty}\left(\frac{A}{\lambda}\right)^{\ell}\sum_{k=1}^{n}\left(\omega^k\right)^{\ell}=n\sum_{j=0}^{\infty}\left(\frac{A^n}{\lambda^n}\right)^{j}$$
$$=n\left[\mathbb{1}-\frac{A^n}{\lambda^n}\right]^{-1}.$$

Therefore the statement holds whenever $|\lambda|>r_A$. To handle $|\lambda|=r_A$, we simply take the limit as $\lambda\to r_A$ in both sides of

$$R_n(\lambda,A)=\left[\mathbb{1}-\frac{A^n}{\lambda^n}\right]^{-1}$$

which gives the desired result. Notice that this is justified as the map $\lambda\mapsto(\lambda\mathbb{1}-A)^{-1}$ is continuous on $r(A)\supseteq S_A$ (where this last inclusion holds by assumption). □

Returning to the main proof, let now $\lambda\in S_A$; a direct calculation verifies that for each $k\in\mathbb{N}$

$$\left\|\left(\mathbb{1}-\frac{\omega^k}{r_A}A\right)^{-1}-\left(\mathbb{1}-\frac{\omega^k}{\lambda}A\right)^{-1}\right\| \tag{7}$$
$$=\left\|\left(\mathbb{1}-\frac{\omega^k}{r_A}A\right)^{-1}\left[\left(\frac{\omega^k}{r}-\frac{\omega^k}{\lambda}\right)A\right]\left(\mathbb{1}-\frac{\omega^k}{\lambda}A\right)^{-1}\right\| \tag{8}$$
$$\leq\left\|\left(\mathbb{1}-\frac{\omega^k}{r_A}A\right)^{-1}\right\|\left|\frac{1}{r}-\frac{1}{\lambda}\right|\|A\|\left\|\left(\mathbb{1}-\frac{\omega^k}{\lambda}A\right)^{-1}\right\| \tag{9}$$
$$=\|A\|\,|\lambda-r_A|\left\|\left(\frac{r_A}{\omega_k}\mathbb{1}-A\right)^{-1}\right\|\left\|(\lambda\mathbb{1}-A)^{-1}\right\| \tag{10}$$
$$\leq\|A\|\,|\lambda-r_A|\sup_{\gamma\in S_A}\left\|(\gamma\mathbb{1}-A)^{-1}\right\|^2. \tag{11}$$

We know that the resolvent function

$$\lambda\mapsto(\lambda\mathbb{1}-A)^{-1}$$

is a continuous function $S_A\subseteq r(A)\to\mathfrak{A}$. Composition with the norm therefore gives another continuous function $S_A\subseteq r(A)\to[0,\infty)$. Now, assume $|\lambda|\geq 1+\|A\|$. From Lemma 1,

$$\left\|(\lambda\mathbb{1}-A)^{-1}\right\|\leq\sum_{n=0}^{\infty}\frac{\|A\|^n}{|\lambda|^{n+1}}=(|\lambda|-\|A\|)^{-1}\leq 1.$$

This means that the map $\lambda \mapsto (\lambda \mathbb{1} - A)^{-1}$ is bounded for $|\lambda| \geq 1 + \|A\|$. On the other hand, notice that

$$\{\lambda \in \mathbb{C} : r_A \leq |\lambda| \leq \|A\| + 1\} \subseteq S_A \subseteq r(A)$$

is compact in $\mathbb{C}$. By continuity, $\lambda \mapsto \left\|(\lambda \mathbb{1} - A)^{-1}\right\|$ is bounded on the compact set

$$\{\lambda \in \mathbb{C} : r_A \leq |\lambda| \leq \|A\| + 1\}$$

and hence on the whole of S_A. Using Lemma 3 together with the inequalities in (7)-(11), we see that for every $n \in \mathbb{N}$

$$\begin{aligned}
&\left\|\left(\mathbb{1} - \frac{A^n}{r_A^n}\right)^{-1} - \left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1}\right\| \\
&= \frac{1}{n}\left\|\sum_{k=1}^{n}\left(\mathbb{1} - \frac{\omega^k}{r_A}A\right)^{-1} - \sum_{k=1}^{n}\left(\mathbb{1} - \frac{\omega^k}{\lambda}A\right)^{-1}\right\| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}\|A\|\,|\lambda - r_A| \sup_{\gamma \in S_A}\left\|(\gamma \mathbb{1} - A)^{-1}\right\|^2 \\
&= \|A\| \sup_{\gamma \in S_A}\left\|(\gamma \mathbb{1} - A)^{-1}\right\|^2 \cdot |\lambda - r_A|
\end{aligned}$$

where $\sup_{\gamma \in S_A}\left\|(\gamma \mathbb{1} - A)^{-1}\right\|^2 < \infty$ by our earlier arguments. Fix $\varepsilon > 0$, from the above we see that we may choose $\lambda \in \mathbb{R}$ with $\lambda > r_A$ such that for all $n \in \mathbb{N}$

$$\left\|\left(\mathbb{1} - \frac{A^n}{r_A^n}\right)^{-1} - \left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1}\right\| < \varepsilon.$$

On the other hand, if $\lambda > r_A$ then

$$\lim_{n \to \infty}\left\|\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1} - \mathbb{1}\right\| = 0.$$

Indeed, $\lambda > r_A$ (with $\lambda \in \mathbb{R}$) implies that $\lambda^m > \|A^m\|$ for all but finitely many m. Hence, repeating arguments used in the proof of Lemma 3, we see that

$$\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1} - \mathbb{1} = \sum_{k=0}^{\infty}\left(\frac{A^n}{\lambda^n}\right)^k - \mathbb{1} = \sum_{k=1}^{\infty}\left(\frac{A^n}{\lambda^n}\right)^k.$$

This implies that

$$\left\|\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1} - \mathbb{1}\right\| \leq \sum_{k=1}^{\infty} \left(\frac{\|A^n\|}{\lambda^n}\right)^k$$

where for all $n \in \mathbb{N}$ sufficiently large we have (by virtue of the absolute convergence in Step 2.1)

$$\frac{\|A^n\|}{\lambda^n} \leq \frac{1}{2}$$

since $\lambda^m > \|A^m\|$ for all but finitely many $m \in \mathbb{N}$. Dominated convergence then implies that

$$\lim_{n\to\infty} \left\|\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1} - \mathbb{1}\right\| = \lim_{n\to\infty} \left\|\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1} - \mathbb{1}\right\| \leq \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\frac{\|A^n\|}{\lambda^n}\right)^k = 0.$$

The triangle inequality gives

$$\left\|\left(\mathbb{1} - \frac{A^n}{r_A^n}\right)^{-1} - \mathbb{1}\right\| \leq \left\|\left(\mathbb{1} - \frac{A^n}{r_A^n}\right)^{-1} - \left(\mathbb{1} - \frac{A^n}{\lambda^n}\right)^{-1}\right\| + \left\|\left(\mathbb{1} - \frac{A^n}{\lambda^n}\right) - \mathbb{1}\right\|$$

Which implies that

$$\limsup_{n\to\infty} \left\|\left(\mathbb{1} - \frac{A^n}{r_A^n}\right)^{-1} - \mathbb{1}\right\| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{n\to\infty} \left\|\left(\mathbb{1} - \frac{A^n}{r_A^n}\right)^{-1} - \mathbb{1}\right\| = 0$$

By our earlier remarks (more precisely, the result in (5)), it follows from the above that

$$\left\|\frac{A^n}{r_A^n}\right\| = \frac{\|A^n\|}{r_A^n} \to 0$$

as $n \to \infty$. As we shall soon see, this is a contradiction. Define a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ by

$$\|A^n\| = \|A\|^{x_n}.$$

This makes sense, for if $\|A^n\| = 0$ for some n then $A^m = 0$ for all $m \geq n$ which contradicts

$$r_A := \limsup_{n\to\infty} \|A^n\|^{1/n} > 0.$$

But then, for all $n, m \in \mathbb{N}$ there holds

$$\|A\|^{x_{n+m}} = \|A^{n+m}\| \leq \|A^n\| \cdot \|A^m\| = \|A\|^{x_n} \cdot \|A\|^{x_m} = \|A\|^{x_n + x_m} .$$

This verifies that $x_{n+m} \leq x_n + x_m$. An application of the Fekete lemma shows that

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf_{n\in\mathbb{N}} \frac{x_n}{n} < \infty.$$

But then,

$$r_A := \lim_{n\to\infty} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A\|^{x_n/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} .$$

This implies that for all $n \in \mathbb{N}$

$$1 \leq \inf_{n\in\mathbb{N}} \frac{\|A^n\|^{1/n}}{r_A} \leq \frac{\|A^n\|^{1/n}}{r_A} \implies 1 \leq \frac{\|A^n\|}{r_A^n}.$$

which contradicts

$$\lim_{n\to\infty} \frac{\|A^n\|}{r_A^n} = 0.$$

This contradiction shows that $S_A \nsubseteq r(A)$. That is, there exists $\lambda \in \mathbb{C}$ with $|\lambda| \geq r_A$ with $\lambda \in \sigma(A)$. In particular, $\sigma(A) \neq \emptyset$. This proves Step 2.2. □

Step 2.3. *The spectral radius formula holds true, i.e.*

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} \leq \|A\| .$$

Proof of Step 2.3. Notice that in Step 2.2 we have shown that

$$\rho(A) \leq \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} \leq \liminf_{n\to\infty} \|A^n\|^{1/n} .$$

Now, later in this step we have shown $\sigma(A)$ to be non-empty by providing the existence of $\lambda \in \sigma(A)$ with $|\lambda| \geq r_A$. This implies that

$$\limsup_{n\to\infty} \|A^n\|^{1/n} = r_A \leq \rho(A) \leq \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} \leq \liminf_{n\to\infty} \|A^n\|^{1/n} .$$

and the theorem immediately follows. ■