ENTIRE NODAL SOLUTIONS WITH PRESCRIBED SYMMETRY TO CAFFARELLI-KOHN-NIRENBERG EQUATIONS

EDWARD CHERNYSH*[1]

[1]McGill University, Canada

ABSTRACT. We establish the existence of sign-changing entire solutions to weighted critical p-Laplace equations of the Caffarelli-Kohn-Nirenberg type. In doing so, we investigate classes of symmetry and show that, for suitable symmetry configurations, there exists a non-trivial solution which changes sign and respects the corresponding prescribed symmetry. In addition, we describe conditions under which these symmetry-types are incompatible. Especially, we demonstrate the existence of entire nodal solutions for an infinite number of distinct symmetry-types.

1. INTRODUCTION

In this article, we investigate the existence and symmetries of sign-changing solutions to the weighted critical equations of the Caffarelli-Kohn-Nirenberg type:

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = |x|^{-bq}\,|u|^{q-2}\,u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0), \end{cases} \tag{1.1}$$

where we assume that $n \geq 4$, $1 < p < n$, and q is the critical Caffarelli-Kohn-Nirenberg (henceforth abbreviated CKN) exponent, and

$$q := \frac{np}{n - p(1 + a - b)}, \quad a < \frac{n-p}{p}. \quad \text{and} \quad a \leq b < a + 1. \tag{1.2}$$

Problem (1.1) is the Euler-Lagrange equation related to the CKN inequality in $\mathbb{R}^n$ (see Caffarelli-Kohn-Nirenberg [3]). The space $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ is the completion of $C^\infty_c(\mathbb{R}^n)$ under the homogenous weighted norm $\|u\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} := \|\nabla u\|_{L^p(\mathbb{R}^n,|x|^{-ap})}$. We refer the reader to the preface of §3, and to Chapter 2 of Chernysh [5], for more details on $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$. We remark that when $a = b = 0$, we simply write $\mathcal{D}^{1,p}(\mathbb{R}^n)$. In this case, (1.1) reduces to the critical p-Laplace equation

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}\,u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^n), \end{cases} \tag{1.3}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$ denotes the p-Laplacian, applied to a function u, and p^* is the critical Sobolev exponent. Should we further restrict to the case $p = 2$, we recover the Yamabe problem, which is itself of important interest.

The existence of nodal solutions to these quasilinear elliptic critical equations has been studied extensively. Addressing the literature on nodal solutions, we first and

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*Corresponding author: Edward Chernysh.

foremost mention the important work of Ding [34], which established the existence of infinitely many sign-changing solutions of finite energy (i.e. solutions belonging to $\mathcal{D}^{1,2}(\mathbb{R}^n)$) when $a = b = 0$ and $p = 2$. Here, Ding exploited the invariance of the unweighted critical Laplace problem under Möbius transformations. We should also take a moment to mention that, for $p = 2$ and $a = b = 0$, other distinct sign-changing solutions were found by Pino-Musso-Pacard-Pistoia [16] by utilizing Lyapunov-Schmidt reduction. Other solutions were found by the same authors in [17]. We refer the reader to the introduction of Clapp-Rios [14] for details on why these approaches do not apply in the case $p \neq 2$ and more information on sign-changing solutions in this broad case. With regards to the case of the sphere, we point to works of Fernández-Palmas-Petean [19], Fernández-Petean [20], and Medina-Musso [25]. For further reference on the literature involving the existence of nodal solutions in the context of the Yamabe problem, we direct the reader to Premoselli-Vétois [27, §2].

With important regard to the critical p-Laplace equation (1.3), we mention that Clapp-Rios [14] demonstrated the existence of sign-changing solutions to problem (1.3) by studying the symmetries of test functions under the action of carefully chosen subgroups of the orthogonal group $O(n)$ (see also Clapp [10] for the case with $p = 2$). An extension of this approach was recently used by Clapp-Faya-Saldaña [13] for the Yamabe equation, and Clapp-Vicente-Benítez [12] to construct solutions exhibiting new symmetries for (1.3) and associated competitive systems. As for weighted equations of the Caffarelli-Kohn-Nirenberg type with subcritical terms, we observe that Szulkin-Waliullah [31] constructed sign-changing solutions when $0 \leq a < b$. Similar approaches have also been used in the fractional case (see, for instance, Hernández-Santamaría-Saldaña [22] and Zhang-Ma-Zheng [36]).

Our aim in this paper is twofold. First, we seek to extend the existence of sign-changing solutions to the weighted problem (1.1) for the full range of parameters (1.2). Second, we investigate distinct types of symmetries one might impose on a sign-changing solution of (1.1), by generalizing the groups constructed by Clapp-Rios [14] and Clapp-Vicente-Benítez [12]. To formally encode these symmetries, we introduce the following notation. Given $n \geq 4$, we write $k = k(n) := \lfloor \frac{n}{2} \rfloor - 1$. We will denote by $\mathfrak{m} = (m_1, \ldots, m_k)$ a tuple of non-negative integers and, for an integer $\alpha \geq 0$, we define the conditions

$$0 < 2\chi_\alpha + \textstyle\sum_{j=1}^{k} m_j(j+1) \leq n/2, \tag{1.4}$$

$$2\chi_\alpha + \textstyle\sum_{j=1}^{k} m_j(j+1) \neq (n-1)/2. \tag{1.5}$$

Here, $\chi_\alpha = 0$ if $\alpha = 0$ and $\chi_\alpha = 1$ otherwise. If $\alpha = 0$, it is required that $\mathfrak{m} \neq \mathbf{0}$. As our approach relies upon the careful construction of symmetry groups, we mention that this first technical condition (1.4) ensures that we are not prescribing symmetries to more coordinate blocks than are available in $\mathbb{R}^n$. On the other hand, the technical requirement (1.5) ensures that every such symmetry group constructed has sufficiently large (i.e. infinite) orbits at all non-zero $x \in \mathbb{R}^n$ with the origin as its only fixed point. This will be a key detail in handling the balanced case $a = b \neq 0$. It should also be noted that (1.5) is the reason for the condition $n \neq 5$ in Theorem 1.1. Indeed, if $n = 5$ then $k = 1$ and (1.4) can only be satisfied for $\alpha = 0$ with and $m_1 = 1$ or $\alpha = 1$ with and $\mathfrak{m} = \mathbf{0}$. However, neither configuration satisfies (1.5).

Roughly speaking, the tuple $\mathfrak{m}$ will encode a precise symmetry configuration describing symmetry under synchronous rotations in specified coordinate tuples, anti-symmetry with respect to reflections and coordinate cycling, and symmetry under isometries of other coordinates. The j^{th} entry m_j of the tuple $\mathfrak{m}$ will define how many distinct actions occur on coordinate tuples of length $2(j+1)$. The parameter $\alpha \geq 0$ dictates the number of "pinwheel-type"-symmetries the function will possess (see also Clapp-Vicente-Benítez [12]). We note that for each index j, these prescribed symmetries act upon distinct and independent coordinate blocks, and condition (1.4) serves only to ensure there are enough coordinates for these symmetries to be well defined. The parameter α is responsible for bringing an infinity of distinctive $\mathfrak{m}$-type symmetries to solutions of (1.1). We refer the reader to §2 for the precise definition of an $(\alpha, \mathfrak{m})$-symmetric function. However, we point out that the solutions of (1.3) obtained in Clapp-Rios [14] are all of $(0, \mathfrak{m})$-type symmetry for $\mathfrak{m} = (j, \mathbf{0})$, with $j \geq 1$. In the case of Clapp-Vicente-Benítez [12], no information is available on the symmetries the solutions satisfy in the last $(n-4)$-coordinates, although the $(\alpha, \mathfrak{m})$-symmetric solutions we produce are certain to include solutions with new symmetries.

Our main result states that there exist solutions with $(\alpha, \mathfrak{m})$-symmetry, corresponding to every pair $(\alpha, \mathfrak{m})$ satisfying (1.4) and, if $a = b \neq 0$, condition (1.5).

Theorem 1.1. *Fix a dimension $n \geq 4$, let $\alpha \geq 0$ be an integer, and $\mathfrak{m}$ be a tuple of non-negative integers such that $(\alpha, \mathfrak{m})$ satisfies (1.4).*

(1) When $a < b$ or $a = b = 0$, there exists a non-trivial sign-changing solution $u_{\alpha,\mathfrak{m}}$ to (1.1) which is $(\alpha, \mathfrak{m})$-symmetric.
(2) When $a = b \neq 0$, $n \neq 5$, and $(\alpha, \mathfrak{m})$ satisfies condition (1.5), there exists a non-trivial sign-changing solution $u_{\alpha,\mathfrak{m}}$ to (1.1) which is $(\alpha, \mathfrak{m})$-symmetric.

Moreover, for any $(\beta, \mathfrak{n})$ satisfying (1.4):

(i) if $\alpha \neq \beta$ then $u_{\alpha,\mathfrak{m}} \neq u_{\beta,\mathfrak{n}}$,
(ii) if $\alpha = \beta$ and $\mathfrak{m} \neq \mathfrak{n}$, then $u_{\alpha,\mathfrak{m}} \neq u_{\beta,\mathfrak{n}}$ provided $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

In this theorem, we are defining for $\mathfrak{m} \neq \mathfrak{n}$ satisfying (1.4)

$$\gcd(\mathfrak{m}, \mathfrak{n}) := \begin{cases} 1 & \text{if } \mathfrak{m} \lesssim \mathfrak{n} \text{ or } \mathfrak{n} \lesssim \mathfrak{m}, \\ \max\{\gcd(j+1, \ell+1) : m_j \neq 0,\, n_\ell \neq 0, \ell \neq j\} & \text{otherwise.} \end{cases} \tag{1.6}$$

In the above, we declare that $\mathfrak{m} \lesssim \mathfrak{n}$ if there exists an index ℓ such that $m_j = n_j$ for all $j < \ell$, $m_\ell \leq n_\ell$, and $m_j = 0$ for all $\ell > j$. Notice that if $\mathfrak{m} \neq \mathfrak{n}$ and the coordinates of $\mathfrak{m}$ and $\mathfrak{n}$ are non-zero only on indices j for which $(j+1)$ is prime, then automatically $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

To the best of the author's knowledge, Theorem 1.1 is the first result giving the existence of entire nodal solutions to the Caffarelli-Kohn-Nirenberg problem (1.1) for the whole range of parameters (1.2). Aside from this important existence question, the main strength of Theorem 1.1 is in its ability to prescribe various mutually exclusive symmetry-types to solutions of (1.1). In fact, by varying $\alpha \geq 0$ for each fixed tuple $\mathfrak{m}$, Theorem 1.1 guarantees the existence of $(\alpha, \mathfrak{m})$-symmetric entire nodal solutions to (1.1) for an infinite number of distinct symmetry configurations. We wish to emphasize that, without some condition such as (i)-(ii) in this last theorem, it is impossible to distinguish the constructed solutions by their symmetries alone. We point the reader towards Remark 2.2 for an instance of a non-trivial function simultaneously satisfying a $(0, (2, \mathbf{0}))$-symmetry and a $(0, (0, 0, 1, \mathbf{0}))$-symmetry. Notice that in this case, both (i) and (ii) fail to apply. As for the solutions constructed in Szulkin-Waliullah [31], we mention that the prescribed symmetries of these solutions are exclusive to two coordinates only.

We also observe that, to the author's knowledge, Theorem 1.1 guarantees the existence of new sign-changing solutions to the critical p-Laplace equation (1.3) and, especially, the Yamabe problem. What is more, for fixed $\alpha \geq 0$, if we restrict ourselves only to $\mathfrak{m}$ such that $m_j \neq 0$ if and only if $(j+1)$ is prime, then we automatically obtain solutions with incompatible symmetries by Theorem 1.1-(ii). In the case of symmetries acting exclusively on a prime number of complex coordinates, the possible choices $S(n)$ of $\mathfrak{m}$ satisfying (1.4)-(1.5) is asymptotically in the order of

$$S(n) \sim \exp\left(\frac{\pi\sqrt{2n}}{\sqrt{3\ln(n/2)}}\right)$$

by Ramanujan [28, Page 260].

Next, we wish to acknowledge the presence of an additional requirement for our construction of a non-trivial nodal solution to (1.1) when $a = b \neq 0$. This requirement is a direct consequence of the particularities present in the bubble tree decomposition for Palais-Smale sequences established in Chernysh [6] for the Caffarelli-Kohn-Nirenberg problem (1.1). It is the nature of this decomposition, together with the inherent lack of translation invariance for (1.1), which contributes this technical requirement. We note that this detail is further elaborated upon in §1.1, where we discuss the proof aspects with greater precision.

Bringing our attention to positive solutions of (1.1), much can be said regarding their classification. Firstly, when $a = b = 0$, all positive solutions to problem (1.1) have been shown to be radially symmetric and an explicit form has been determined. This was demonstrated by Caffarelli, Gidas and Spruck [2] in the case $p = 2$, by Damascelli, Merchán, Montoro and Sciunzi [15] for $\frac{2n}{n+2} \leq p < 2$, by Vétois [32] for $1 < p < 2$, and finally by Sciunzi [29] for $2 < p < n$. We also observe that significant work has been put forth in the direction of extending this classification to positive solutions of (1.3), without the assumption that the solutions lie in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. Indeed, we point to the papers of Catino-Monticelli-Roncoroni [4], Ou [26], Vétois [33] and Sun-Wang [30]. Recently, Ciraolo-Gatti [8] established such a classification for positive local solutions of (1.3), under certain boundedness or growth conditions.

In the weighted setting (i.e. when $(a, b) \neq (0, 0)$), the classification of positive solutions to (1.1) has been established in certain special cases. For instance, when $b \neq a = 0$, Ghoussoub-Yuan [21] determined an explicit form for every positive radial solution of (1.1). In the case $p = 2$, Dolbeault-Esteban [18] provided an explicit form (demonstrating, in particular, radial symmetry) for every positive solution to (1.1) in the non-limiting case $a < b$, provided a and b satisfy certain conditions. In dimensions $n \geq 3$, Ciraolo-Corso [7] extended this classification to include all $1 < p < n$ when $a = b \geq 0$, as well as a wider range of possible values for a, b provided $p < n/2$. Additionally, Le-Le [24] later showed that, when $a = 0$, all positive solutions of (1.1) are radial and therefore classified by Ghoussoub-Yuan [21]. With regards to classifying positive local solutions of (1.1) (e.g. positive solutions which, a priori, need not live in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$), we direct the reader to the paper of Ciraolo-Polvara [9]. We refer the reader to §1.1.1 of [6] for further historical context regarding the classification of solutions to (1.1).

1.1. **Notation and Proof Discussions.** Throughout this paper, G denotes a closed subgroup of $O(n)$. Given a point $x \in \mathbb{R}^n$, we shall write $\mathrm{Orb}_G(x)$ and $\mathrm{Stab}_G(x)$ to denote the orbit of x and stabilizer of x, respectively, under the action of G. That is, given $x \in \mathbb{R}^n$ we define

$$\mathrm{Orb}_G(x) := \{gx : g \in G\} \quad \text{and} \quad \mathrm{Stab}_G(x) := \{g \in G : gx = x\}.$$

A subset $\Omega \subseteq \mathbb{R}^n$ is said to be G-*invariant* provided

$$\mathrm{Orb}_G(\Omega) := \bigcup_{x \in \Omega} \mathrm{Orb}_G(x) \subseteq \Omega.$$

Of course, this is equivalent to requiring that $\mathrm{Orb}_G(\Omega) = \Omega$. If Ω is a G-invariant domain, a function $u : \Omega \to \mathbb{R}$ will be called G-invariant if

$$u(gx) = u(x), \quad \forall g \in G, \, x \in \Omega.$$

For the remainder of this paper, whenever a subgroup G of $O(n)$ is present, $\Omega \subseteq \mathbb{R}^n$ will be assumed to be G-invariant containing the origin.

Given a subgroup G of $O(n)$, a continuous group homomorphism $\phi : G \to \{\pm 1\}$, and a G-invariant domain $\Omega \subseteq \mathbb{R}^n$, a test function $u \in C_c^\infty(\Omega)$ will be called ϕ-equivariant if

$$u(gx) = \phi(g) u(x), \quad \forall g \in G, \, x \in \Omega.$$

We shall denote by $C_c^\infty(\Omega)^\phi$ the vector space consisting of all ϕ-equivariant functions in $C_c^\infty(\Omega)$. For the proof of Theorem 1.1, we construct suitable subgroups G of $O(n)$ along with continuous group homomorphisms $\phi : G \to \{\pm 1\}$ such that the following important properties hold.

(P1) There exists $\xi \in \mathbb{R}^n$ such that $\mathrm{Stab}_G(\xi) \subseteq \ker \phi$.
(P2) ϕ is surjective.

These two properties follow those employed within Clapp-Rios [14]. Property (P1) is crucial to the argument as it ensures that the space of ϕ-equivariant test functions $C_c^\infty(\Omega)^\phi$ is non-trivial, and hence infinite dimensional; see Bracho-Clapp-Marzantowicz [1, page 195]. For the limit case $a = b \neq 0$ in the problem parameters, we must impose the following additional constraint on the subgroup G:

(P3) For each $x \in \mathbb{R}^n \setminus \{0\}$, the orbit $\mathrm{Orb}_G(x)$ is infinite.

Condition (P3) is a subtle, and yet important, distinction from that present in Clapp-Rios [14]. Indeed, within this paper, the authors allow the subgroup G to satisfy either $\dim(\mathrm{Orb}_G(x)) > 0$ or $\mathrm{Orb}_G(x) = \{x\}$, for every $x \in \mathbb{R}^n$. However, within the weighted case with $a = b \neq 0$, it is crucial that we ensure the orbit $\mathrm{Orb}_G(x)$ be infinite for $x \neq 0$. This is because, by the Struwe-type decomposition for weighted critical p-Laplace equations from [6], solutions to (1.1) when $a = b \neq 0$ are produced precisely when concentration occurs at the origin at a sufficient rate (see [6, Theorem 2]). Consequently, in order to ensure that our process yields solutions to the weighted problem in $\mathbb{R}^n$, we must enforce concentration at the origin by requiring that the origin be the only point fixed by the action of G. Without the strengthened condition (P3), there is the possibility that our procedure produces solutions to an unweighted problem, either in $\mathbb{R}^n$ or a half-space. In sharp contrast, however, when $a < b$, all concentration must automatically take place at the origin (see [6, Theorem 1]) which renders condition (P3) superfluous, thereby allowing us to construct more solutions.

The remainder of this paper is delegated to the proof of Theorem 1.1. In the next section, we construct suitable subgroups G of $O(n)$ along with associated homomorphisms $G \to \{\pm 1\}$. Our construction generalizes that of Clapp-Rios [14] and Clapp-Vicente-Benítez [12], most notably by considering combinations of group actions on complex tuples whose length is any integer larger than 1, rather than only actions on pairs of complex coordinates. Following this, we prove that solutions may be distinguished by their symmetry-types, i.e. we establish the second part of Theorem 1.1. To achieve this, we utilize a binary coding scheme to encode the coordinates with respect to which our equivariant solutions are rotationally invariant in order to deduce that these solutions satisfy mutually exclusive symmetry-types.

Afterwards, in §3, we show that symmetric Palais-Smale sequences may be constructed at suitable energy levels which we shall exploit to demonstrate the existence of sign-changing solutions. Using these sequences, we employ a modified symmetrized analogue of the Struwe-type decomposition for weighted critical p-Laplace equations to extract a non-trivial solution to a symmetrized version of (1.1). We then show that this solution also solves the problem (1.1) itself.

2. Constructing Admissible Symmetry Groups

In this section, we construct the subgroups of $O(n)$ needed to produce the symmetric Palais-Smale sequences required within §3-4 of this paper. As stated in the introduction, our groups generalize those obtained by Clapp-Rios [14] and Clapp-Vicente-Benítez [12] by incorporating actions involving non-trivial tuples of complex coordinates of arbitrary length.

Henceforth, we fix a dimension $n \geq 4$ and put $k = k(n) := \lfloor n/2 \rfloor - 1$.

2.1. **Building Groups of Prescribed Symmetry.** For each integer $j \geq 1$, we define an $\mathbb{R}$-linear isometry of $\mathbb{R}^{2(j+1)} \cong \mathbb{C}^{j+1}$ given by

$$\rho_j : \mathbb{C}^{j+1} \to \mathbb{C}^{j+1}, \quad (z_1, \ldots, z_{j+1}) \mapsto (-\overline{z_{j+1}}, \overline{z_1}, \ldots, \overline{z_j}). \tag{2.1}$$

Clearly, ρ_j^ℓ contains a non-trivial permutation of the coordinates $(z_1, \ldots, z_{j+1})$ unless $\ell \equiv 0 \mod (j+1)$. Furthermore, when $j+1$ is even, there holds for $z \in \mathbb{C}^{j+1}$,

$$\rho_j^{j+1}(z) = -z \quad \text{and} \quad \rho_j^{2(j+1)}(z) = z$$

On the other hand, it is easily seen that if $j+1$ is odd:

$$\rho_j^{j+1}(z_1, \ldots, z_{j+1}) = (-\bar{z}_1, -\bar{z}_2, \ldots, -\overline{z_{j+1}}) \quad \text{and} \quad \rho_j^{2(j+1)}(z) = z.$$

Regardless, ρ_j has order $2(j+1)$ for each $j \geq 1$. Additionally, we allow rotations to act upon the coordinates of $\mathbb{C}^{j+1}$ in the synchronous coordinate-wise manner:

$$e^{i\theta}(z_1, \ldots, z_{j+1}) := (e^{i\theta} z_1, \ldots, e^{i\theta} z_{j+1}),$$

for $\theta \in [0, 2\pi)$. Define Γ_j to be the subgroup of $O(2(j+1))$, acting upon the coordinates of $\mathbb{R}^{2(j+1)} \cong \mathbb{C}^{j+1}$, generated by these rotations and the map ρ_j. We emphasize that, when $j = 1$, this idea comes from Clapp [10] and Clapp-Rios [14].

An easy calculation shows that, for each $\theta \in [0, 2\pi)$,

$$\begin{aligned} e^{i\theta}\rho_j(z_1, \ldots, z_{j+1}) = e^{i\theta}(-\overline{z_{j+1}}, \bar{z}_1, \ldots, \overline{z_j}) &= (-e^{i\theta}\overline{z_{j+1}}, e^{i\theta}\bar{z}_1, \ldots, e^{i\theta}\overline{z_{jj}}) \\ &= (-\overline{e^{-i\theta} z_{j+1}}, \overline{e^{-i\theta} z_1}, \ldots, \overline{e^{-i\theta} z_j}) \\ &= \rho_j e^{i(-\theta)}(z_1, \ldots, z_{j+1}) \\ &= \rho_j e^{i(2\pi-\theta)}(z_1, \ldots, z_{j+1}). \end{aligned}$$

Consequently, rotations permute with ρ_j up to a change in angle in the sense that $e^{i\theta}\rho_j = \rho_j e^{i(2\pi-\theta)}$. In the event that $j+1$ is even, we have $\rho_j^{j+1}(z) = -z$ whence $\rho_j^{j+1} = e^{i\pi}$ is a rotation. However, for all $(j+1)$-odd, no composition of ρ_j can be realized as a rotation (except the identity). It follows from these observations that

Γ_j consists precisely of those linear isometries $g \in O(2(j+1))$ uniquely expressible in the form

$$\begin{cases} g = \rho_j^{\varepsilon} e^{i\theta}, & \varepsilon \in \{0, 1, \ldots, j\},\ \theta \in [0, 2\pi) \quad \text{if } j+1 \text{ is even,} \\ g = \rho_j^{\varepsilon} e^{i\theta}, & \varepsilon \in \{0, 1, \ldots, 2j+1\},\ \theta \in [0, 2\pi) \quad \text{if } j+1 \text{ is odd.} \end{cases} \tag{2.2}$$

This representation allows us to define a mapping

$$\phi_j : \Gamma_j \to \{\pm 1\}, \quad g \mapsto (-1)^{\varepsilon},$$

where g is expressed uniquely as in (2.2); clearly, this is a group homomorphism. Moreover, ϕ_j is continuous being that it is a homomorphism of topological groups that is continuous at the identity. Indeed, consider $\eta_j := (1+i, 1, 0 \ldots, 0) \in \mathbb{C}^{j+1}$ and write $g \in \Gamma_j$ according to (2.2). If $(j+1)$ is even and $\varepsilon > 0$, then $|g\eta_j - \eta_j| \geq 1$ implying that any g sufficiently close to the identity belongs to the kernel of ϕ_j. Similarly, when $(j+1)$ is odd, we clearly have $|g\eta_j - \eta_j| \geq 1$ unless $\varepsilon = 0$ or $\varepsilon = j+1$. When $\varepsilon = j+1$, direct computation shows that

$$|g\eta_j - \eta_j|^2 = 6 - 4\sin(\theta) + 2\cos(\theta) \geq 6 - 2\sqrt{5}.$$

Thus, again in this case, any $g \in \Gamma_j$ sufficiently close to the identity lies in the kernel of ϕ_j. This demonstrates the continuity of ϕ_j on Γ_j.

2.1.1. *Incorporating Pinwheel-type Symmetries.* In the following, we borrow ideas from Clapp-Vicente-Benítez [12] (see also Clapp-Faya-Saldanã [13], and Clapp-Saldaña-Soares-Benítez [11]) in order to produce solutions satisfying a "pinwheel-type" symmetry within the weighted case. We remark that whether or not we incorporate such a symmetry is encoded by the value of α from Theorem 1.1. More precisely, α will determine the number of "pins" within the symmetry class of our solution.

As in the paper of Clapp-Vicente-Benítez [12], we begin by considering an action on $\mathbb{C}^2 \cong \mathbb{R}^4$. We allow the unit circle to act upon $\mathbb{C}^2$ via asynchronous rotation:

$$e^{i\theta}(z_1, z_2) := (e^{i\theta} z_1, e^{-i\theta} z_2).$$

Next, given a non-negative integer α and $\beta \in \mathbb{N}$, we adopt the transformation

$$\varrho_{\alpha}^{\beta} : \mathbb{C}^2 \to \mathbb{C}^2, \quad z \mapsto \cos\left(\frac{\beta\pi}{2^{\alpha+1}}\right) z + \sin\left(\frac{\beta\pi}{2^{\alpha+1}}\right) \rho_1(z).$$

For convenience, we write $\varrho_{\alpha} := \varrho_{\alpha}^1$. As illustrated in Clapp-Vicente-Benítez [12], ϱ_{α} commutes with these rotations and, moreover, ϱ_{α} has order $2^{\alpha+2}$. In fact, there holds $\varrho_{\alpha}^{\beta}\varrho_{\alpha}^{\beta'} = \varrho_{\alpha}^{\beta+\beta'}$. If $\alpha = 0$, let

$$\Upsilon_{\alpha} = \{I\}$$

where I denotes the identity of $O(n)$. Otherwise, let Υ_{α} denote the copy in $O(n)$ of the $O(4)$-subgroup generated by ϱ_{α} and these rotations. Next, by the relations above, the mapping

$$\varphi_{\alpha} : \Upsilon_{\alpha} \to \{\pm 1\}$$

induced by the rules $\varphi_{\alpha}(e^{i\theta}) := 1$ and $\varphi_{\alpha}(\varrho_{\alpha}) := -1$ is a well defined group homomorphism. If $\alpha = 0$, we let φ_{α} be the trivial homomorphism. Notice (see, for instance, Clapp-Vicente-Benítez [12]) that Υ_{α} is compact in $O(n)$ and φ_{α} is continuous. Furthermore, it is easy to see that the point $(1, 0) \in \mathbb{C}^2$ satisfies the condition $\mathrm{Stab}_{\Upsilon_{\alpha}}(1, 0) \subseteq \ker \varphi_{\alpha}$.

2.2. **Group Construction.** Fix now a dimension $n \geq 4$ and a non-negative integer α. Suppose we are given a tuple $\mathfrak{m} := (m_1, \ldots, m_k)$ of non-negative integers and define the following conditions:

Assume henceforth that (1.4) is satisfied. For each index $j = 1, \ldots, k$, we denote by $\Gamma_{j,\ell}$, with $\ell = 1, \ldots, m_j$, the copy of Γ_j in $O(n)$ acting upon the coordinates of $\mathbb{R}^n$ indexed by

$$\begin{cases} \sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2(\ell-1)(j+1) + 5 & \text{if } \alpha > 0, \\ \sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2(\ell-1)(j+1) + 1 & \text{if } \alpha = 0, \end{cases}$$

through

$$\begin{cases} \sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2\ell(j+1) + 4 & \text{if } \alpha > 0, \\ \sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2\ell(j+1) & \text{if } \alpha = 0. \end{cases}$$

Since $\Gamma_{j,\ell}$ is an isomorphic copy of Γ_j, we may abuse notation and view ϕ_j as a group homomorphism $\Gamma_{j,\ell} \to \{\pm 1\}$ in the natural way.

According to this scheme, given any non-negative integer α and $\mathfrak{m}$ such that $(\alpha, \mathfrak{m})$ satisfies (1.4), we define a subgroup $G_{\alpha,\mathfrak{m}}$ of $O(n)$ by putting

$$G_{\alpha,\mathfrak{m}} := \Upsilon_\alpha \Lambda_{\alpha,\mathfrak{m}} \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}, \tag{2.3}$$

where $\Lambda_{\alpha,\mathfrak{m}}$ is a copy of the orthogonal group:

$$\Lambda_{\alpha,\mathfrak{m}} \cong \begin{cases} O\left(n - 4 - 2\sum_{j=1}^{k} m_j(j+1)\right) & \text{if } \alpha > 0 \text{ and (1.5) holds,} \\ O\left(n - 2\sum_{j=1}^{k} m_j(j+1)\right) & \text{if } \alpha = 0 \text{ and (1.5) holds,} \\ \{I\} & \text{if (1.5) fails} \end{cases}$$

acting on the remaining coordinates of $\mathbb{R}^n$, untouched by all of the $\Gamma_{j,\ell}$ and Υ_α. Observe that $G_{\alpha,\mathfrak{m}}$ is indeed a well defined subgroup of $O(n)$ since it is the product of subgroups of $O(n)$, with each subgroup commuting with the others.

It is apparent from the definition in (2.3) that each $g \in G_{\alpha,\mathfrak{m}}$ may be uniquely expressed as a product of commutative factors belonging to the respective subgroups appearing in the definition of $G_{\alpha,\mathfrak{m}}$, in the sense that

$$g = \upsilon_\alpha g_0 \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \gamma_{j,\ell}, \quad \upsilon_\alpha \in \Upsilon_\alpha,\ g_0 \in \Lambda_{\alpha,\mathfrak{m}},\ \gamma_{j,\ell} \in \Gamma_{j,\ell}. \tag{2.4}$$

Consequently, we define a group homomorphism $\phi_{\alpha,\mathfrak{m}} : G_{\alpha,\mathfrak{m}} \to \{\pm 1\}$ by allowing each ϕ_j to act on the factors of g belonging to the copies of Γ_j, i.e.

$$\phi_{\alpha,\mathfrak{m}}(g) := \varphi_\alpha(\upsilon_\alpha) \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \phi_j(\gamma_{j,\ell}),$$

where g is expressed as in (2.4).

Before we proceed with some essential lemmas regarding the properties of the pairs $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,\mathfrak{m}})$ constructed above, let us formally encode some notation which is utilized in the statement Theorem 1.1.

Definition 2.1. Maintaining the aforedescribed notation, a function $u : \mathbb{R}^n \to \mathbb{R}$ will be said to be $(\alpha, \mathfrak{m})$-symmetric if it is $\phi_{\alpha,\mathfrak{m}}$-equivariant.

Roughly speaking, an $\mathfrak{m}$-type symmetry describes in which ways the solution u, with respect to the remaining coordinates not affected by Υ_α, may be synchronously rotated, reflected, and coordinate swapped in relation to the function's sign. Furthermore, as we shall later demonstrate, these indeed give rise to distinct and incompatible symmetries, in contexts where the assumptions stated in the second part of Theorem 1.1 hold.

Remark 2.2. It is natural to ask whether one must impose some condition on $\mathfrak{m}$ and $\mathfrak{n}$, as in Theorem 1.1-(ii), to distinguish the solutions when $\alpha = \beta$. Without any additional condition, there is no way, in general, to distinguish these solutions using the symmetries alone. For instance, the mapping

$$\mathbb{R}^8 \cong \mathbb{C}^4 \to \mathbb{R}, \quad z \mapsto \Im\left(z_1\overline{z_2}z_3\overline{z_4}\right)$$

is an example of a non-trivial function which is both $(0,(2,\mathbf{0}))$-symmetric and $(0,(0,0,1,\mathbf{0}))$-symmetric. Here, $\Im(z)$ denotes the imaginary part of a complex number z. Consequently, if we allow for actions on a composite number of complex coordinates, there is no way, in general, to distinguish the resulting symmetry-types. In addition, to show that our imposed symmetries are mutually exclusive, we utilize a binary coding scheme together with a gcd-argument requiring that we compare symmetries on a coprime number coordinates. We refer the reader to Lemmas 2.5 and 2.6, as well as Proposition 2.7 for the concrete arguments.

Next, we show that the group and homomorphism pairs constructed above satisfy (P1)-(P2) whenever condition (1.4) holds. In addition, we verify that (1.5) implies property (P3).

Lemma 2.3. *Let $n \geq 4$, $\alpha \geq 0$ an integer, and $\mathfrak{m} := (m_1, \dots, m_k)$ be a tuple of non-negative integers such that $(\alpha, \mathfrak{m})$ satisfies (1.4). Then, the group $G_{\alpha,\mathfrak{m}}$ defined in (2.3) is a closed subgroup of $O(n)$, and $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,\mathfrak{m}})$ satisfies conditions (P1)-(P2). Furthermore, if $(\alpha, \mathfrak{m})$ satisfies (1.5), then (P3) holds true.*

Proof. We only explicitly treat the case $\alpha > 0$, since a direct adaptation of this argument applies in the special case $\alpha = 0$ (see also the end of this proof for comments on how the proof is adapted in this case).

First we demonstrate that $G_{\alpha,\mathfrak{m}}$ is a closed subgroup of $O(n)$. Indeed, owing to the fact that each ρ_j has finite order, and the set of actions by rotation form a continuous copy of the unit circle in $O(2(j+1))$, it is easy to see that Γ_j is compact in $O(2(j+1))$. It follows that each $\Gamma_{j,\ell}$ is compact in $O(n)$. By Tychonoff's theorem and continuity of the group operation on $O(n)$, we infer that $\Upsilon_\alpha \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}$ is compact in $O(n)$. Then, since $\Lambda_{\alpha,\mathfrak{m}}$ is closed in $O(n)$, it follows from standard theory on topological groups (see Hewitt-Ross [23]) that $G_{\alpha,\mathfrak{m}}$ is closed in $O(n)$, which verifies our first assertion.

By construction, $\phi_{\alpha,\mathfrak{m}}$ is surjective and hence criterion (P2) holds true. In addition, it is easy to see that $\phi_{\alpha,\mathfrak{m}}$ is continuous on $G_{\alpha,\mathfrak{m}}$ since, by our discussions in §2.1-§2.1.1, φ_α and each ϕ_j are continuous. To see that the property (P1) holds, let $\mathbf{e} = (1, 0) \in \mathbb{C}^2$ and define for each index $j = 1, \dots, k$ the point $\xi_j = (1 + i, 1, \dots, 1) \in \mathbb{C}^{j+1}$; consider

$$\xi = (\mathbf{e}, \xi_1, \dots, \xi_1, \dots, \xi_k, \dots, \xi_k, w) \in \mathbb{C}^2 \times \left(\prod_{j=1}^{k} \mathbb{C}^{m_j(j+1)}\right) \times \mathbb{R}^{n-4-2\sum_{j=1}^{k} m_j(j+1)},$$

with $w \in \mathbb{R}^{n-4-2\sum_{j=1}^{k} m_j(j+1)}$ arbitrary. Above, we emphasize that each ξ_j appears exactly m_j-times in the tuple defining ξ. We claim that $\mathrm{Stab}_{G_{\alpha,\mathfrak{m}}}(\xi) \subseteq \ker \phi_{\alpha,\mathfrak{m}}$.

Certainly, express any $g \in \mathrm{Stab}_{G_{\alpha,\mathfrak{m}}}(\xi)$ according to (2.4) and observe that $g\xi = \xi$ forces $\upsilon_\alpha \mathbf{e} = \mathbf{e}$ and $\gamma_{j,\ell}\xi_j = \xi_j$ for all $j = 1, \dots, k$ and $\ell = 1, \dots, m_j$. By our comments at the end of §2.1.1, we have $\upsilon_\alpha \in \ker \varphi_\alpha$. For each admissible pair j, ℓ, write $\gamma_{j,\ell}$ in the form of (2.2). That is, $\gamma_{j,\ell} = \rho_{j,\ell}^{\varepsilon_{j,\ell}} e^{i\theta_{j,\ell}}$ with $\varepsilon_{j,\ell} \in \{0, 1, \dots, j\}$ if $j+1$ is even and $\varepsilon_{j,\ell} \in \{0, 1, \dots, 2j+1\}$ otherwise. Here, $\rho_{j,\ell}$ denotes the copy of ρ_j acting upon the coordinates treated by $\Gamma_{j,\ell} \cong \Gamma_j$. Note that, because $\rho_{j,\ell}^{\varepsilon_{j,\ell}}$ shuffles coordinates unless $\varepsilon_{j,\ell} \equiv 0 \mod (j+1)$, we must have either $\varepsilon_{j,\ell} = 0$ or $\varepsilon_{j,\ell} = j+1$. If $j+1$ is even, the only possibility is $\varepsilon_{j,\ell} = 0$. If $j+1$ is odd, then in the event that $\varepsilon_{j,\ell} = j+1$, the equality $\rho_{j,\ell}^{\varepsilon_{j,\ell}} e^{i\theta_{j,\ell}} \xi_j = \xi_j$ reduces to

$$\left(-e^{-i\theta_{j,\ell}}(1-i), -e^{-i\theta_{j,\ell}}, -e^{-i\theta_{j,\ell}}, \dots, -e^{-i\theta_{j,\ell}}\right) = (1+i, 1, 1, \dots, 1)$$

which admits no possible solutions. Consequently, each $\varepsilon_{j,\ell} = 0$ whence $g \in \ker \phi_{\alpha,\mathfrak{m}}$ by construction. This verifies (P1).

It remains to establish (P3) when $(\alpha, \mathfrak{m})$ satisfies the additional condition (1.5). To this end, notice that for any

$$x = (z, \zeta_{1,1}, \dots, \zeta_{1,m_1}, \dots, \zeta_{k,1}, \dots, \zeta_{k,m_k}, w) \in \mathbb{C}^2 \times \left(\prod_{j=1}^k \mathbb{C}^{m_j(j+1)}\right) \times \mathbb{R}^{n-4-2\sum_{j=1}^k m_j(j+1)}$$

there holds

$$\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x) = \Upsilon_\alpha z \times \left(\prod_{j=1}^k \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}\zeta_{j,\ell}\right) \times \Lambda_{\alpha,\mathfrak{m}} w. \tag{2.5}$$

Thus, so long as $x \neq 0$, the cardinality of $\mathrm{Orb}_G(x)$ shall be infinite because condition (1.5) ensures that $\Lambda_{\alpha,\mathfrak{m}}$ is infinite in the case $n \neq 4 + 2\sum_{j=1}^k m_j(j+1)$. This demonstrates (P3) and concludes the proof.

In the case $\alpha = 0$, we proceed in the same way as above. Indeed, by the same reasoning as before, for $\alpha = 0$ and $\mathfrak{m}$ satisfying (1.4), the group $G_{\alpha,\mathfrak{m}}$ given by (2.3) defines a closed subgroup of $O(n)$. The corresponding homomorphism is obviously surjective by construction. To verify the stabilizer condition (P1), we use the same argument as in the case $\alpha = 0$, but omitting the point $\mathbf{e} \in \mathbb{C}^2$ and only including the required copies of $\xi_j \in \mathbb{C}^{j+1}$ within the definition of ξ. If $(\alpha, \mathfrak{m})$ satisfies (1.4)-(1.5) with $\alpha = 0$, then the orbit of a point $x = (\zeta_{1,1}, \dots, \zeta_{1,m_1}, \dots, \zeta_{k,1}, \dots, \zeta_{k,m_k}, w)$ is given by

$$\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x) = \left(\prod_{j=1}^k \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}\zeta_{j,\ell}\right) \times \Lambda_{\alpha,\mathfrak{m}} w,$$

and hence (P3) holds true by the same logic as for the case $\alpha > 0$. □

2.3. **The Distinguishing of Symmetry Configurations.** We now turn towards the task of investigating when the symmetry-types $(\alpha, \mathfrak{m})$ satisfying (1.4) are incompatible. To achieve this, we leverage a structure which we call a t-code.

For any positive integer t, given a binary vector $c := (c_1, \dots, c_t) \in \mathbb{Z}_2^t$, we shall write $\tilde{c}$ to denote the *cycle* of c given by

$$\tilde{c} := (c_t, c_1, \dots, c_{t-1}).$$

Definition 2.4. A t-code is a subset $\mathcal{C}$ of the vector space $\mathbb{Z}_2^t$ such that

(1) If $c, c' \in C$ with $c \leq c'$ (meaning $c_j \leq c'_j$ for all $j = 1, \dots, t$) then $c + c' \in \mathcal{C}$.
(2) $\tilde{c} \in \mathcal{C}$ for each $c \in \mathcal{C}$.

An element $c \in \mathcal{C}$ is called a *codeword* (of $\mathcal{C}$).

Given an arbitrary integer $0 \leq r \leq t$, it will be convenient to associate it with the codeword

$$v_r := (1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \mathbb{Z}_2^t,$$

whose first r-coordinates are 1 and the remaining $(t-r)$-coordinates are 0. Note that v_0 is simply the zero codeword while v_t is the codeword consisting entirely of ones.

Before presenting the main lemma of this subsection, we make a simple observation. If $r \leq s \leq t$ are non-negative integers and $v_r, v_s \in \mathcal{C}$ where $\mathcal{C}$ is a t-code, then $v_{s-r} \in \mathcal{C}$. Indeed, if $v_r, v_s \in \mathcal{C}$ then $v_r + v_s \in \mathcal{C}$ and repeatedly cycling $v_r + v_s$ yields $v_{s-r} \in \mathcal{C}$.

Lemma 2.5. *Let $r < s \leq t$ be positive integers such that $\gcd(r, s) = 1$. Suppose that $\mathcal{C}$ is a t-code containing both v_r and v_s. Then, $\mathcal{C}$ contains the standard basis of $\mathbb{Z}_2^t$.*

Proof. Since $\mathcal{C}$ is closed under cycling, it suffices to show that $v_1 = (1, 0, \ldots, 0) \in \mathcal{C}$. If $r = 1$ there is nothing to show. Otherwise, we write

$$s = k_1 r + r_1$$

for a non-negative integer k_1 and an integer remainder $0 < r_1 < r$. Then, since $v_r, v_s \in \mathcal{C}$ we see that $v_{s-k_1 r} = v_{r_1} \in \mathcal{C}$. If $r_1 = 1$ then we are done. Otherwise, we continue and write

$$r = k_2 r_1 + r_2$$

for a non-negative integer k_2 and an integer remainder $0 < r_2 < r_1$. Since $v_r \in \mathcal{C}$ and $v_{r_1} \in \mathcal{C}$, we also have that $v_{r-k_2 r_1} = v_{r_2} \in \mathcal{C}$. Again, if $r_2 = 1$ then we are done. Otherwise, continuing this process, since $\gcd(r, s) = 1$ the Euclidean algorithm guarantees that, eventually, we have a remainder of 1. So $v_1 \in \mathcal{C}$ and we are done. □

2.4. **Rotation Codewords.** Keeping with the notation presented in this section, let $\alpha \geq 0$ be an integer and $\mathfrak{m}$ a tuple such that $(\alpha, \mathfrak{m})$ satisfies (1.4). Let $1 \leq j \leq k$ be such that $m_j > 0$. Fix an index $\ell \in \{1, \ldots, m_j\}$ and denote by $(z_1, z_2, \ldots, z_{j+1})$ the (complex) coordinates the group $\Gamma_{j,\ell} \subseteq G_{\alpha,\mathfrak{m}}$ acts upon. Then, any $x \in \mathbb{R}^n$ may be expressed as $x = (w_1, z_1, \ldots, z_{j+1}, w_2)$, where w_1 and w_2 are real tuples containing, respectively, the coordinates of x before and after $(z_1, \ldots, z_{j+1})$.

Given a binary codeword $c = (c_1, \ldots, c_{j+1}) \in \mathbb{Z}_2^{j+1}$ of length $(j+1)$, we define a rotation operator

$$R_c^{\alpha,\mathfrak{m},j,\ell}(\theta) : \mathbb{R}^n \to \mathbb{R}^n, \quad (w_1, z_1, \ldots, z_{j+1}, w_2) \mapsto (w_1, e^{c_1 i\theta} z_1, \ldots, e^{c_{j+1} i\theta} z_{j+1}, w_2).$$

The key observation is that the set of all $c \in \mathbb{Z}_2^{j+1}$ such that u is invariant under $R_c^{\alpha,\mathfrak{m},j,\ell}(\theta)$ is itself a $(j+1)$-code.

Lemma 2.6. *Let $n \geq 4$ and $(\alpha, \mathfrak{m})$ be a pair for which (1.4) holds. Suppose $1 \leq j \leq k$ is an index such that $m_j > 0$ and fix $\ell \in \{1, \ldots, m_j\}$. Assume u is $(\alpha, \mathfrak{m})$-symmetric and let $\mathcal{C}$ be the set of all binary codewords $c \in \mathbb{Z}_2^{j+1}$ such that*

$$u(R_c^{\alpha,\mathfrak{m},j,\ell}(\theta)x) = u(x)$$

for all $x \in \mathbb{R}^n$ and $\theta \in [0, 2\pi)$. Then, $\mathcal{C}$ is a $(j+1)$-code.

Proof. For concision, let us write $R_c(\theta) := R_c^{\alpha,\mathfrak{m},j,\ell}(\theta)$ for the duration of the proof. Clearly, $\mathcal{C}$ contains $(0,0,\dots,0)$ by definition. If $c, c' \in \mathcal{C}$ with $c \leq c'$ then $c + c' \in \mathcal{C}$ since

$$\begin{aligned} R_c(-\theta)R_{c'}(\theta)(w_1, z_1, \dots, z_{j+1}, w_2) &= (w_1, e^{(c_1'-c_1)i\theta}z_1, \dots, e^{(c_{j+1}'-c_{j+1})i\theta}z_{j+1}, w_2) \\ &= R_{c+c'}(\theta)(w_1, z_1, \dots, z_{j+1}, w_2) \end{aligned}$$

where $c + c'$ is computed in $\mathbb{Z}_2^{j+1}$. Next, to show $\mathcal{C}$ is closed under cycling, we write

$$\begin{aligned} \rho_j R_c(-\theta)\rho_j^{-1}(w_1, z_1, \dots, z_{j+1}, w_2) &= \rho_j R_c(-\theta)(w_1, \overline{z_2}, \dots, \overline{z_{j+1}}, -\overline{z_1}, w_2) \\ &= \rho_j(w_1, e^{-i\theta c_1}\overline{z_2}, \dots, e^{-i\theta c_j}\overline{z_{j+1}}, -e^{-i\theta c_{j+1}}\overline{z_1}, w_2) \\ &= (w_1, e^{i\theta c_{j+1}}z_1, e^{i\theta c_1}z_2, \dots, e^{i\theta c_j}z_{j+1}, w_2) \\ &= R_{\tilde{c}}(\theta)x. \end{aligned}$$

In the above, $\rho_j = \rho_{j,\ell}$ is the copy of $\rho_j \in \Gamma_j$ appearing in $\Gamma_{j,\ell}$. Thus, $\tilde{c} \in \mathcal{C}$ and we are done. □

Proposition 2.7. *Fix a dimension $n \geq 4$. Let $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ be pairs such that each satisfies condition (1.4). Assume that $u : \mathbb{R}^n \to \mathbb{R}$ is $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric. If $\alpha = \beta$ and $\mathfrak{m} \neq \mathfrak{n}$, then u vanishes identically if* $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

Proof. We may assume without harm to the proof that $\alpha = \beta = 0$ (otherwise we consider only the coordinates $(x_5, x_6, \dots, x_n)$ of x). By hypothesis, $\mathfrak{m} \neq \mathfrak{n}$. Thus, there exists a minimal index $j \in \{1, \dots, k\}$ such that $n_j \neq m_j$. Without loss of generality, suppose we are in the case $m_j < n_j$. We now distinguish two possible settings.

(1) If $m_\ell = 0$ for all indices $\ell > j$ then, owing to the fact that $m_j < n_j$, there exists a copy of ρ_j acting on $2(j+1)$-coordinates such that

$$u(\rho_j x) = u(x) \quad \text{and} \quad u(\rho_j x) = -u(x), \quad \forall x \in \mathbb{R}^n,$$

where the first identity arises from the $\phi_{\alpha,\mathfrak{m}}$-equivariance of u and the second comes from the $\phi_{\beta,\mathfrak{n}}$-equivariance. We conclude that $u \equiv 0$.

(2) Otherwise, let $\ell > j$ be the minimal index for which $m_\ell > 0$. In this case, there exists a tuple consisting of $2(\ell+1)$-coordinates on which a copy of Γ_ℓ acts, but for which the first $2(j+1)$-coordinates are acted upon by a copy of Γ_j.

Let now $\mathcal{C}$ denote the set of all binary codewords $c \in \mathbb{Z}_2^{\ell+1}$ such that

$$u(R_c^{\alpha,\mathfrak{m},\ell,1}(\theta)x) = u(x), \quad \forall \theta \in [0, 2\pi)$$

and all $x \in \mathbb{R}^n$. By Lemma 2.6, $\mathcal{C}$ is a $(\ell+1)$-code. From the isomorphic copies of Γ_ℓ and Γ_j, it is automatic that $v_{j+1}, v_{\ell+1} \in \mathcal{C}$. Observe that the condition $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$ enforces $\gcd(\ell+1, j+1) = 1$. An application of Lemma 2.5 implies that $\mathcal{C}$ contains the standard basis of $\mathbb{Z}_2^{\ell+1}$, thereby ensuring that u is invariant under rotations in each individual (complex) coordinate acted upon by the aforementioned copy of Γ_ℓ.

By the aforementioned gcd-condition, at least one of $\ell+1$ or $j+1$ must be odd. Thus, we are reduced to two cases.

(a) If $(\ell+1)$ is odd, then

$$\begin{aligned} u(w_1, z_1, \dots, z_{\ell+1}, w_2) &= -u(\rho_\ell^{\ell+1}(w_1, z_1, \dots, z_{\ell+1}, w_2)) \\ &= -u(w_1, -\overline{z_1}, \dots, -\overline{z_{\ell+1}}, w_2) \end{aligned}$$

for any $(w_1, z_1, \dots, z_{\ell+1}, w_2) \in \mathbb{R}^n$. But then, since u is invariant under rotations in each of the (complex) coordinates $z_1, \dots, z_{\ell+1}$, it follows that

$$u(w_1, -\overline{z_1}, \dots, -\overline{z_{\ell+1}}, w_2) = u(w_1, z_1, \dots, z_{\ell+1}, w_2).$$

Combining the last two equations, we get

$$u(w_1, z_1, \dots, z_{\ell+1}, w_2) = -u(w_1, z_1, \dots, z_{\ell+1}, w_2).$$

We conclude that $u \equiv 0$.

(b) If $(j+1)$ is odd then, by the same argument,

$$\begin{aligned} u(w_1, z_1, \dots, z_{j+1}, w_2) &= -u(\rho_j^{j+1}(w_1, z_1, \dots, z_{j+1}, w_2)) \\ &= -u(w_1, -\overline{z_1}, \dots, -\overline{z_{j+1}}, w_2) \end{aligned}$$

for any $(w_1, z_1, \dots, z_{\ell+1}, w_2) \in \mathbb{R}^n$. Using that u is rotation invariant in each of the coordinates $z_1, \dots, z_{j+1}, \dots, z_{\ell+1}$, we again find that

$$u(w_1, -\overline{z_1}, \dots, -\overline{z_{j+1}}, w_2) = u(w_1, z_1, \dots, z_{j+1}, w_2).$$

As before, we obtain

$$u(w_1, z_1, \dots, z_{j+1}, w_2) = -u(w_1, z_1, \dots, z_{j+1}, w_2).$$

Ergo, $u \equiv 0$.

In any case, we deduce that $u \equiv 0$ which completes the proof. □

Proposition 2.8. *Let $n \geq 4$ and suppose $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ are two tuples, each satisfying (1.4). Let $u : \mathbb{R}^n \to \mathbb{R}$ be $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric. If $\alpha \neq \beta$, then u vanishes identically.*

Proof. Since $\alpha \neq \beta$, we may assume without loss of generality that $\alpha < \beta$. In this case, we must distinguish between $\alpha = 0$ and $\alpha > 0$. In the latter case, we have $\varrho_\alpha = \varrho_\beta^{2^{\beta-\alpha}}$ implying that, for all $x \in \mathbb{R}^n$,

$$u(x) = u(\varrho_\beta^{2^{\beta-\alpha}}(x)) = u(\varrho_\alpha(x)) = -u(x),$$

where the first equality uses the $(\beta, \mathfrak{n})$-symmetry of u and the final equality invokes the $(\alpha, \mathfrak{m})$-symmetry. Consequently, we infer that $u \equiv 0$.

If instead $\alpha = 0$, then observing that $\varrho_\beta^{2^\beta} = \rho_1$ yields a similar contradiction provided $m_1 \neq 0$. Indeed,

$$u(x) = u(\varrho_\beta^{2^\beta}(x)) = u(\rho_1(x)) = -u(x)$$

whence $u \equiv 0$. It remains to treat the setting where $0 = \alpha < \beta$ and $m_1 = 0$. In this case, let $j > 1$ be minimal such that $m_j \neq 0$. Denote by $(z_1, z_2, \dots, z_{j+1})$ the complex coordinates acted upon by the first copy of Γ_j in $G_{\alpha,\mathfrak{m}}$. Then, for any $x = (z_1, \dots, z_{j+1}, w) \in \mathbb{R}^n \cong \mathbb{C}^{j+1} \times \mathbb{R}^{n-2(j+1)}$, there holds

$$u(x) = u(e^{i\theta} z_1, \dots, e^{i\theta} z_{j+1}, w) \tag{2.6}$$

for any $\theta \in \mathbb{R}$, by virtue of the $(\alpha, \mathfrak{m})$-symmetry. Owing to the $(\beta, \mathfrak{n})$-symmetry, we also have

$$u(x) = u(e^{i\theta} z_1, e^{-i\theta} z_2, z_3, \dots, z_{j+1}, w)$$

for all angles $\theta \in \mathbb{R}$. By following the argument in Lemma 2.6 (i.e. conjugating the above rotations by the copy of ρ_j in $G_{\alpha,\mathfrak{m}}$ acting on the coordinates $(z_1,\dots,z_{j+1})$), we see that

$$
\begin{aligned}
u(x) &= u(e^{i\theta}z_1, e^{-i\theta}z_2, z_3, \dots, z_j, z_{j+1}, w) && \text{(E.1)}\\
&= u(z_1, e^{i\theta}z_2, e^{-i\theta}z_3, z_4, \dots, z_j, z_{j+1}, w) && \text{(E.2)}\\
&= u(z_1, z_2, e^{i\theta}z_3, e^{-i\theta}z_4, \dots, z_j, z_{j+1}, w) && \text{(E.3)}\\
&\ \ \vdots\\
&= u(z_1, z_2, z_3, z_4, \dots, e^{i\theta}z_j, e^{-i\theta}z_{j+1}, w) && \text{(E.j)}
\end{aligned}
$$

Now, given θ, we sequentially apply (E.1) with $\theta/(j+1)$, (E.2) with $2\theta/(j+1)$, (E.3) with $3\theta/(j+1)$, and so forth, until we apply (E.j) with $j\theta/(j+1)$. Each such rotation preserves the sign of u, whence

$$
u(x) = u(e^{i\theta/(j+1)}z_1, e^{i\theta/(j+1)}z_2, \dots, e^{i\theta/(j+1)}z_j, e^{-ij\theta/(j+1)}z_{j+1}, w).
$$

Then, applying (2.6) with $-\theta/(j+1)$, we obtain

$$
u(x) = u(z_1, z_2, \dots, z_j, e^{-i\theta}z_{j+1}, w).
$$

Consequently, u is rotation invariant in the complex coordinate z_{j+1}. Following once more the conjugation argument from Lemma 2.6, it follows that u is rotation invariant in each complex coordinate of $(z_1,\dots,z_{j+1})$.

Next, by using that u is $(\beta,\mathfrak{n})$-symmetric with $\beta > 0$, we obtain for every point $x = (z,w) \in \mathbb{R}^n \cong \mathbb{C}^{j+1} \times \mathbb{R}^{n-2(j+1)}$ that

$$
\begin{aligned}
u(x) = u(\varrho_\beta^{2^\beta}(x)) &= u(-\bar{z}_2, \bar{z}_1, z_3, z_4, \dots, z_j, z_{j+1}, w)\\
&= u(z_2, z_1, z_3, z_4, \dots, z_j, z_{j+1}, w),
\end{aligned}
$$

where we have used that u is invariant under rotation in each coordinate of z for this last step. Again, by conjugation with respect to ρ_j, the same argument as before (see also Lemma 2.6) implies that any two adjacent coordinates of $z = (z_1,\dots,z_{j+1})$ may be swapped while preserving the value of $u(z)$. Iterating this principle yields the identity

$$
u(x) = u(z_{j+1}, z_1, z_2, \dots, z_j, w). \tag{2.7}
$$

On the other hand, the $(\alpha,\mathfrak{m})$-symmetry of u together with the rotational invariance in each coordinate of z ensures that

$$
\begin{aligned}
-u(x) = u(\rho_j(x)) &= u(-\overline{z_{j+1}}, \bar{z}_1, \dots, \bar{z}_j, w)\\
&= u(z_{j+1}, z_1, \dots, z_j, w).
\end{aligned}
$$

Combining this last equation with (2.7), we conclude that $u(x) = 0$.

In all cases we have shown that $u \equiv 0$, which is what had to be shown. □

Next, we aim to verify that the symmetry incompatibilities we have established in Propositions 2.7-2.8 also ensure that the solutions we will produce in Theorem 1.1 remain distinct even with regards to translations and dilations. This is of important relevance to our problem (1.1) since, if a function u is a solution to (1.1), then so is the dilated function $x \mapsto \lambda^\gamma u(\lambda x)$, for $\lambda > 0$, where $\gamma > 0$ is the homogeneity exponent given by

$$
\gamma := \frac{n - bq}{q} = \frac{n - p(1+a)}{p}. \tag{2.8}
$$

If $a = b = 0$, then translation is allowed in the sense that

$$x \mapsto \lambda^\gamma u(\lambda x + y), \quad y \in \mathbb{R}^n,$$

is also a solution to (1.1). As a first step in this direction, we show that a non-trivial solution of (1.1) with $(\alpha, \mathfrak{m})$-symmetry cannot be a translation of a $(\beta, \mathfrak{n})$-symmetric solution, provided $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ satisfy one of the conditions dictated by the second part of Theorem 1.1.

Lemma 2.9. *Fix a dimension $n \geq 4$. Suppose that $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ are two pairs, each of which satisfies condition (1.4). Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be non-trivial solutions to (1.1) which are $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric, respectively. Assume that one of the following conditions hold:*

(1) $\alpha \neq \beta$;
(2) $\alpha = \beta$ and $\mathfrak{m} \neq \mathfrak{n}$ with $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

Then, u is not a translation of v.

Proof. Suppose for a contradiction that there exists a non-zero $x_0 \in \mathbb{R}^n$ for which there holds $u(x) = v(x - x_0)$ for all $x \in \mathbb{R}^n$. We distinguish two cases.

(1) If both $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ satisfy (1.5), then the transformation

$$\varsigma : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto -x$$

satisfies the rules

$$u(\varsigma(x)) = u(x) \quad \text{and} \quad v(\varsigma(x)) = v(x)$$

for all $x \in \mathbb{R}^n$. Notice that ς is $\mathbb{R}$-linear. Consequently, for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} u(-x) = u(x) = v(x - x_0) = v(\varsigma(x - x_0)) &= v(-x + x_0) \\ &= u(-x + 2x_0). \end{aligned}$$

As $x \in \mathbb{R}^n$ was arbitrary, we infer that u is periodic and non-trivial; this contradicts u being a non-vanishing solution of (1.1).

(2) Consider next the case where $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ do not both satisfy (1.5), and assume without loss of generality that $(\alpha, \mathfrak{m})$ fails this condition. In particular, n must be odd. Then, we consider the transformation

$$\varsigma : \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \ldots, x_{n-1}, x_n) \mapsto (-x_1, \ldots, -x_{n-1}, x_n).$$

Again, $\varsigma \in G_{\alpha,\mathfrak{m}}$ is an $\mathbb{R}$-linear isometry of $\mathbb{R}^n$. Furthermore, $u(\varsigma x) = u(x)$ for all $x \in \mathbb{R}^n$ because ς is composed of sign-preserving components of $G_{\alpha,\mathfrak{m}}$. In fact, we can say the same for v because

(a) If $(\beta, \mathfrak{n})$ does not satisfy (1.5) we get $v(\varsigma x) = v(x)$ by the same reasoning as for u.
(b) If $(\beta, \mathfrak{n})$ satisfies (1.5) there is a non-trivial copy of an orthogonal group acting upon the final coordinates not touched by any sign-changing action from $G_{\beta,\mathfrak{n}}$ whence ς is a sign-preserving transformation in $G_{\beta,\mathfrak{n}}$ with respect to $\phi_{\beta,\mathfrak{n}}$.

Now, $\varsigma^2 = I$ and so

$$\begin{aligned} u(x) = u(\varsigma(x)) = v(\varsigma(x) - x_0) &= v(x - \varsigma(x_0)) \\ &= v(x - \varsigma(x_0) + x_0 - x_0) \\ &= u(x - \varsigma(x_0) + x_0). \end{aligned}$$

Hence, we have periodicity of u provided $\varsigma(x_0) \neq x_0$. Of course, by the same reasoning as earlier, this would be a contradiction. If $\varsigma(x_0) = x_0$, this means that all but the last coordinate of x_0 is 0, at which point we have $\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x_0) = \{x_0\}$ by (1.5). But then, $v(x) = u(x + x_0)$ is a translation of u by a $G_{\alpha,\mathfrak{m}}$-invariant point in space. Thus, v is $\phi_{\alpha,\mathfrak{m}}$-equivariant and must be trivial by Propositions 2.7-2.8.

□

The next corollary verifies the distinctness of our solutions under the previously mentioned translations and dilations.

Corollary 2.10. *Fix a dimension $n \geq 4$. Suppose that $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ are two pairs, each of which satisfies condition (1.4). Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be non-trivial solutions to (1.1) which are $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric, respectively. Assume one of the following hold true:*

(1) $\alpha \neq \beta$;
(2) $\alpha = \beta$ and $\mathfrak{m} \neq \mathfrak{n}$ with $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

Then, u is not a translated rescaling of v, in the sense that there does not exist $\lambda > 0$ and $y \in \mathbb{R}^n$ such that

$$u(x) = \lambda^\gamma v(\lambda x + y).$$

Here, $\gamma > 0$ is the homogeneity exponent given by (2.8).

Proof. By way of contradiction, suppose we can write

$$u(x) = \lambda^\gamma v(\lambda x + y)$$

for some $\lambda > 0$ and $y \in \mathbb{R}^n$. Since u is non-trivial, we must have $\lambda \neq 0$. If $y = 0$, then $u(x) = w(x)$ where $w(x) := \lambda^\gamma v(\lambda x)$ is itself $\phi_{\beta,\mathfrak{n}}$-equivariant. Notice also that w is a solution of (1.1). Consequently, Propositions 2.7-2.8 imply that u is trivial which contradicts our assumptions. If instead $y \neq 0$, we observe that

$$u(x) = \lambda^\gamma v(\lambda x + y) = \lambda^\gamma v\left(\lambda\left[x + \frac{y}{\lambda}\right]\right) = w\left(x + \frac{y}{\lambda}\right),$$

meaning that u is a translation of a $\phi_{\beta,\mathfrak{n}}$-equivariant solution of (1.1). Ergo, we are in contradiction with Lemma 2.9. □

3. Existence of Symmetric Palais-Smale Sequences

In this section, we construct Palais-Smale sequences at suitable energy levels for the extraction of non-trivial solutions to (1.1). Although our approach here closely mimics that found in Clapp-Rios [14], we include the details here for the sake of completeness due to the presence of weights and the modified group conditions.

Recall that, given an open set $\Omega \subseteq \mathbb{R}^n$ containing the origin, we let $\mathcal{D}_a^{1,p}(U, 0)$ denote the weighted homogenous Sobolev space obtained by the completion of $C_c^\infty(\Omega)$ under the weighted norm

$$\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} := \|\nabla u\|_{L^p(\Omega,|x|^{-ap})} = \left(\int_\Omega |\nabla u|^p \, |x|^{-ap} \, \mathrm{d}x\right)^{1/p}.$$

As mentioned briefly within the introduction, the Sobolev space $\mathcal{D}_a^{1,p}(\Omega, 0)$ possesses the expected properties; we refer the reader to [5, Chapter 2] for more detail.

The energy functional associated to the weighted critical p-Laplace problem in (1.1) is the C^1-functional $J : \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p \, |x|^{-ap} \, \mathrm{d}x - \frac{1}{q} \int_{\mathbb{R}^n} |u|^q \, |x|^{-bq} \, \mathrm{d}x \, .$$

A function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ is said to be a solution to problem (1.1) if the Fréchet derivative of J vanishes on $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ at u, i.e. if

$$\langle J'(u), h \rangle = 0, \quad \forall h \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$$

where $\langle \cdot, * \rangle$ denotes the duality pairing and

$$\langle J'(u), h \rangle = \int_{\mathbb{R}^n} |\nabla u|^{p-2} \, \nabla u \cdot \nabla h \, |x|^{-ap} \, \mathrm{d}x - \int_{\mathbb{R}^n} |u|^{q-2} \, uh \, |x|^{-bq} \, \mathrm{d}x.$$

Recall that, given a Banach space V together with a C^1-functional E on V, a Palais-Smale sequence (henceforth abbreviated (PS)-sequence) for E is a sequence (v_α) in V such that

(1) $E(v_\alpha)$ converges as $\alpha \to \infty$,
(2) $E'(v_\alpha) \to 0$ in the strong operator topology of the dual space V^*.

The limit value $\lim_{\alpha\to\infty} E(v_\alpha)$ is often called the *energy level* of the (PS)-sequence.

Henceforth throughout this section, unless stated otherwise, G will denote a closed subgroup of $O(n)$, $\Omega \subseteq \mathbb{R}^n$ a G-invariant domain containing the origin, and $\phi : G \to \{\pm 1\}$ a continuous homomorphism of groups satisfying (P1).

We define $\mathcal{D}_a^{1,p}(\Omega, 0)^\phi$ to be the topological closure of $C_c^\infty(\Omega)^\phi$ within the weighted homogeneous Sobolev space $\mathcal{D}_a^{1,p}(\Omega, 0)$. Naturally, as the space $\mathcal{D}_a^{1,p}(\Omega, 0)$ is a reflexive Banach space for $1 < p < n$, the symmetrized space $\mathcal{D}_a^{1,p}(\Omega, 0)^\phi$ inherits these properties. We refer the reader to [5, Chapter 2] and [6] for a more detailed description of $\mathcal{D}_a^{1,p}(\Omega, 0)$ and for analogues of well known foundational results from classical Sobolev theory (e.g. standard embedding theorems, Rellich-Kondrachov, and Riesz-Representation results). As an intermediary step for Theorem 1.1, we shall be forced to consider solutions to a "symmetrized" version of problem (1.1) related to the subgroups G and corresponding homomorphisms ϕ constructed in the previous section. Formally, we shall say that $u \in \mathcal{D}_a^{1,p}(\Omega, 0)^\phi$ is a solution to

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap} \, |\nabla u|^{p-2} \, \nabla u\right) = |x|^{-bq} \, |u|^{q-2} \, u \quad \text{in } \Omega, \\ u \in \mathcal{D}_a^{1,p}(\Omega, 0)^\phi, \end{cases} \tag{3.1}$$

provided $\langle J'(u), h \rangle = 0$ for all $h \in \mathcal{D}_a^{1,p}(\Omega, 0)^\phi$. We also point out that, by extension by 0 outside Ω, it is automatic that $\mathcal{D}_a^{1,p}(\Omega, 0)^\phi \subseteq \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)^\phi \subseteq \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$.

We shall also explicitly introduce an entire analogue of this symmetrized problem in $\mathbb{R}^n$. Namely, we say that a function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)^\phi$ is a solution to

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap} \, |\nabla u|^{p-2} \, \nabla u\right) = |x|^{-bq} \, |u|^{q-2} \, u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)^\phi, \end{cases} \tag{3.2}$$

if $\langle J'(u), h \rangle = 0$ for all $h \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)^\phi$.

Definition 3.1. We define $\mathcal{N}^\phi(\Omega)$ to be the Nehari manifold

$$\mathcal{N}^\phi(\Omega) := \left\{ u \in \mathcal{D}_a^{1,p}(\Omega, 0)^\phi \setminus \{0\} : \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u\|^q_{L^q(\Omega, |x|^{-bq})} \right\}.$$

Subsequently, we set

$$c^{\phi}(\Omega) := \inf_{u\in\mathcal{N}^{\phi}(\Omega)} J(u) = \inf_{u\in\mathcal{N}^{\phi}(\Omega)} \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} |\nabla u|^{p} |x|^{-ap} \, \mathrm{d}x \geq 0. \tag{3.3}$$

By testing a solution u of (3.1) against itself, it is easy to see that the Nehari manifold $\mathcal{N}^{\phi}(\Omega)$ contains all non-trivial solutions of (3.1), should one exist. We also point out that $\mathcal{N}^{\phi}(\Omega)$ is non-empty; this is verified in this next lemma which compiles several important properties relating to this Nehari manifold and the minimal energy level $c^{\phi}(\Omega)$. We emphasize that the results in this section are standard (see, for instance, Clapp-Rios [14] and Clapp-Vicente-Benítez [12]), but choose to include certain details for the sake of completeness.

Lemma 3.2. *There hold the following.*

(1) *The Nehari manifold $\mathcal{N}^{\phi}(\Omega)$ is non-empty.*
(2) *There exists $\mu_0 > 0$ such that $\|u\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \geq \mu_0$ for all $u \in \mathcal{N}^{\phi}(\Omega)$. Consequently, $c^{\phi}(\Omega) > 0$.*
(3) *We have*

$$c^{\phi}(\Omega) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

where $\Gamma \subseteq C\left([0,1], \mathcal{D}^{1,p}_a(\Omega,0)^{\phi}\right)$ consists of those continuous paths

$$\gamma : [0,1] \to \mathcal{D}^{1,p}_a(\Omega,0)^{\phi}$$

beginning at 0 with non-trivial endpoint of non-positive energy, i.e.

$$\Gamma := \left\{\gamma \in C\left([0,1], \mathcal{D}^{1,p}_a(\Omega,0)^{\phi}\right) : \gamma(0) = 0,\ \gamma(1) \neq 0,\ J(\gamma(1)) \leq 0\right\}.$$

(4) *$C^{\infty}_c(\Omega)^{\phi} \cap \mathcal{N}^{\phi}(\Omega)$ is dense in $\mathcal{N}^{\phi}(\Omega)$.*
(5) *The Nehari manifold $\mathcal{N}^{\phi}(\Omega)$ is a closed C^1-Banach submanifold of $\mathcal{D}^{1,p}_a(\Omega,0)^{\phi}$, and a natural constraint for the energy functional J.*

Proof.

(1) Certainly, since $C^{\infty}_c(\Omega)^{\phi}$ is non-trivial (it is infinite dimensional) we may choose $u \in C^{\infty}_c(\Omega)^{\phi}$ such that $u \neq 0$. Consider the mapping

$$\ell : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|tu\|^{p}_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|tu\|^{q}_{L^{q}(\Omega,|x|^{-bq})}$$

which is clearly continuous. By inspection, it is easy to see that $\ell(t) > 0$ for t near 0 whilst $\ell(t) < 0$ for all $|t|$ sufficiently large. By the Intermediate Value Theorem, it follows that $\ell(t_0 u) = 0$ for some $t_0 \neq 0$ whence $t_0 u \in \mathcal{N}^{\phi}(\Omega)$ since, of course, $t_0 u \neq 0$.

(2) By the CKN inequality, given $u \in \mathcal{N}^{\phi}(\Omega)$, we have

$$\begin{aligned} 0 &= \|u\|^{p}_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|u\|^{q}_{L^{q}(\Omega,|x|^{-bq})} \\ &\geq \|u\|^{p}_{\mathcal{D}^{1,p}_a(\Omega,0)} - S\,\|u\|^{q}_{\mathcal{D}^{1,p}_a(\Omega,0)} \\ &= \|u\|^{p}_{\mathcal{D}^{1,p}_a(\Omega,0)} \left(1 - S\,\|u\|^{q-p}_{\mathcal{D}^{1,p}_a(\Omega,0)}\right) \end{aligned}$$

for a suitable constant $S > 0$. Since $q > p$, this is only possible if the $\mathcal{D}^{1,p}_a(\Omega,0)$-norms of $u \in \mathcal{N}^{\phi}(\Omega)$ cannot be made arbitrarily close to 0. Thus, there exists $\mu_0 > 0$ such that $\|u\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \geq \mu_0$ for all $u \in \mathcal{N}^{\phi}(\Omega)$, as was asserted.

(3) First, given $u \in \mathcal{N}^\phi(\Omega)$ we have that $u \neq 0$ and $\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} = \|u\|^q_{L^q(\Omega,|x|^{-bq})}$ whence, for $t > 1$ sufficiently large,

$$\begin{aligned} J(tu) &= \frac{\|tu\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}{p} - \frac{\|tu\|^q_{L^q(\Omega,|x|^{-bq})}}{q} \\ &= \left(\frac{t^p}{p} - \frac{t^q}{q}\right) \|u\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \\ &< 0. \end{aligned}$$

Furthermore, by the first derivative test, it is easy to see that $J(tu)$ (with $t > 0$) achieves its maximum at $t = 1$. Let $s_0 > 1$ be such that $J(s_0 u) < 0$ and consider the path

$$\gamma_0 : [0,1] \to \mathcal{D}^{1,p}_a(\Omega,0)^\phi$$

given by $\gamma_0(t) := ts_0 u$. Then, $\gamma_0(0) = 0$, $\gamma_0(1) = s_0 u \neq 0$, and $J(\gamma_0(1)) < 0$. Thus, $\gamma_0 \in \Gamma$. Observing also that

$$\max_{t\in[0,1]} J(\gamma_0(t)) = \max_{t\in[0,1]} J(ts_0 u) = J(u),$$

we infer that

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \leq \max_{t\in[0,1]} J(\gamma_0(t)) = J(u).$$

Since $u \in \mathcal{N}^\phi(\Omega)$ was arbitrary, it follows that

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \leq \inf_{u\in N^\phi(\Omega)} J(u) = c^\phi(\Omega).$$

To establish the reverse inequality, consider the function

$$\kappa : \mathcal{D}^{1,p}_a(\Omega,0) \to \mathbb{R}, \quad u \mapsto \begin{cases} \dfrac{\|u\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

By appealing to the CKN inequality, this mapping is easily seen to be continuous: indeed, for all $u \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi \setminus \{0\}$ we have

$$|\kappa(u)| = \frac{\|u\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}} \leq S \|u\|^{q-p}_{\mathcal{D}^{1,p}_a(\Omega,0)}.$$

Obviously, $\kappa(0) = 0$ while, if $u \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi$ has non-positive energy (i.e. if $J(u) \leq 0$), then

$$\kappa(u) = \frac{\|u\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}} \geq \frac{q}{p} > 1.$$

Consequently, for all paths $\gamma \in \Gamma$, we have $\kappa(\gamma(0)) = 0$ and $\kappa(\gamma(1)) > 1$ so that (by the Intermediate Value Theorem) there exists $t_1 \in (0,1)$ with the property that $\kappa(\gamma(t_1)) = 1$. Then, $\gamma(t_1) \in \mathcal{N}^\phi(\Omega)$ and

$$\max_{t\in[0,1]} J(\gamma(t)) \geq J(\gamma(t_1)) \geq c^\phi(\Omega).$$

As this holds for all paths $\gamma \in \Gamma$, we deduce that

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq c^\phi(\Omega),$$

whence
$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) = c^{\phi}(\Omega),$$
as was asserted.

(4) For this point, let $u \in \mathcal{N}^{\phi}(\Omega)$ be given. Then, $u \neq 0$, $u \in \mathcal{D}^{1,p}_a(\Omega,0)^{\phi}$, and $\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} = \|u\|^p_{L^q(\Omega,|x|^{-bq})}$. Since $C^\infty_c(\Omega)^{\phi}$ is dense in $\mathcal{D}^{1,p}_a(\Omega,0)^{\phi}$ by definition, we may select a sequence (u_α) in $C^\infty_c(\Omega)^{\phi}$ such that $u_\alpha \to u$ in $\mathcal{D}^{1,p}_a(\Omega,0)$, as $\alpha\to\infty$. Without loss of generality, we may assume that each $u_\alpha \neq 0$. For each index $\alpha \in \mathbb{N}$, we define
$$t_\alpha := \left(\frac{\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}}\right)^{\frac{1}{q-p}} > 0$$
and set $v_\alpha := t_\alpha u_\alpha$. Then, $v_\alpha \in C^\infty_c(\Omega)^{\phi} \subset \mathcal{D}^{1,p}_a(\Omega,0)^{\phi}$ for each $\alpha\in\mathbb{N}$ and $v_\alpha \in \mathcal{N}^{\phi}(\Omega)$ since
$$\begin{aligned}
\|v_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} &= \|t_\alpha u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&= t^p_\alpha \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&= \left[\frac{\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}}\right]^{\frac{p}{q-p}} \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&= \left[\frac{\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}}\right]^{\frac{q}{q-p}} \left[\frac{\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}}\right]^{\frac{p-q}{q-p}} \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&= t^q_\alpha \left[\frac{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}\right]^{\frac{q-p}{q-p}} \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&= t^q_\alpha \|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})} \\
&= \|v_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}.
\end{aligned}$$
In addition, by continuity of the $\mathcal{D}^{1,p}_a(\Omega,0)$-norm and the CKN inequality, it follows from $u_\alpha\to u$ in $\mathcal{D}^{1,p}_a(\Omega,0)$ that $t_\alpha \to 1$, as $\alpha\to\infty$. Consequently,
$$\begin{aligned}
\|v_\alpha - u\|_{\mathcal{D}^{1,p}_a(\Omega,0)} &\leq \|v_\alpha - u_\alpha\|_{\mathcal{D}^{1,p}_a(\Omega,0)} + \|u_\alpha - u\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&= \|t_\alpha u_\alpha - u_\alpha\|_{\mathcal{D}^{1,p}_a(\Omega,0)} + o(1) \\
&= (t_\alpha - 1)\|u_\alpha\|_{\mathcal{D}^{1,p}_a(\Omega,0)} + o(1) \\
&= o(1)
\end{aligned}$$
whence we infer that $v_\alpha \to u$ strongly in $\mathcal{D}^{1,p}_a(\Omega,0)$.

(5) This follows by a verbatim extension of the argument used within Clapp-Rios [14, Lemma 2.3]. ☐

As an important point, we now verify that this minimal energy level $c^{\phi}(\Omega)$ is independent of our choice of G-invariant domain Ω. It is precisely this energy invariance that will ultimately allow us to conclude that our extracted function indeed solves the entire problem posed in (1.1).

Lemma 3.3. *We have $c^{\phi}(\Omega) = c^{\phi}$, where we define $c^{\phi} := c^{\phi}(\mathbb{R}^n)$.*

Proof. Because $\mathcal{D}^{1,p}_a(\Omega,0)^{\phi} \subseteq \mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi}$ after an extension by 0 outside Ω, it is automatic that $c^{\phi} \leq c^{\phi}(\Omega)$. In order to establish the reverse inequality, let

$u \in \mathcal{N}^\phi(\mathbb{R}^n)$ and select a sequence (u_α) in $C_c^\infty(\mathbb{R}^n)^\phi \cap \mathcal{N}^\phi(\mathbb{R}^n)$ converging to u in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ as $\alpha \to \infty$. Since each u_α has compact support and Ω contains the origin, we may select for each index $\alpha \in \mathbb{N}$ some $\lambda_\alpha > 0$ such that the rescaled function

$$v_\alpha(x) := \lambda_\alpha^{-\gamma} u_\alpha\left(\frac{x}{\lambda_\alpha}\right)$$

is compactly supported in Ω. Here, $\gamma > 0$ is the homogeneity exponent, associated to the CKN-inequality, defined in (2.8). By this homogeneity property, $\|v_\alpha\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} = \|u_\alpha\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}$ and $\|u_\alpha\|_{L^q(\mathbb{R}^n,0)} = \|v_\alpha\|_{L^q(\mathbb{R}^n,0)}$ which implies that $J(v_\alpha) = J(u_\alpha)$ and $v_\alpha \in \mathcal{N}^\phi(\Omega)$. Finally, observe that

$$c^\phi(\Omega) \leq J(v_\alpha), \quad \forall \alpha \in \mathbb{N}$$

whence $c^\phi(\Omega) \leq \lim_{\alpha\to\infty} J(v_\alpha) = \lim_{\alpha\to\infty} J(u_\alpha) = J(u)$. As $u \in \mathcal{N}^\phi(\mathbb{R}^n)$ is arbitrary, it follows that $c^\phi(\Omega) \leq c^\phi$. □

Corollary 3.4 (Existence of (PS)-Sequence at the minimal energy level c^ϕ)**.** *There exists a (PS)-sequence in $\mathcal{D}_a^{1,p}(\Omega, 0)^\phi$ at energy level c^ϕ. Namely, there exists a sequence (u_α) in $\mathcal{D}_a^{1,p}(\Omega, 0)^\phi$ such that*

$$\lim_{\alpha\to\infty} J(u_\alpha) = c^\phi \quad \text{and} \quad J'(u_\alpha) \to 0 \text{ in } \mathcal{D}_a^{-1,p'}(\Omega, 0)^\phi.$$

Proof. By Lemma 3.3, it suffices to construct such a (PS)-sequence at the energy level $c^\phi(\Omega)$. Letting Γ be as in Lemma 3.2, notice that given any path $\gamma \in \Gamma$, the composition $t \mapsto J(\gamma(t))$ is a continuous mapping $[0, 1] \to \mathbb{R}$ and is therefore bounded. Consequently,

$$c^\phi(\Omega) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) < \infty.$$

Next, there exists by Lemma 3.2 a constant $\mu_0 > 0$ such that for all $u \in \mathcal{N}^\phi(\Omega)$ there holds $\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \geq \mu_0$. It follows that

$$\begin{aligned} J(u) &= \frac{\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)}}{p} - \frac{\|u\|_{L^q(\Omega,|x|^{-bq})}}{q} \\ &= \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}\left(\frac{1}{p} - \frac{1}{q}\right) \\ &\geq \mu_0^p\left(\frac{1}{p} - \frac{1}{q}\right) \\ &> 0 \end{aligned}$$

uniformly over $u \in \mathcal{N}^\phi(\Omega)$. Thus,

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) = c^\phi(\Omega) \geq \mu_0^p\left(\frac{1}{p} - \frac{1}{q}\right) > 0$$

whilst $J(\gamma(0)) = 0$ and $J(\gamma(1)) \leq 0$ for all $\gamma \in \Gamma$. Consequently, the assertion follows after an application of Theorems 2.8-2.9 from Willem [35]. □

4. Palais-Smale Symmetric Criticality

For this section, $G \subseteq O(n)$ is a closed subgroup and $\phi : G \to \{\pm 1\}$ a continuous group homomorphism such that (P1) holds. Moreover, we let $\Omega \subseteq \mathbb{R}^n$ be a G-invariant domain containing the origin. The next proposition asserts that Palais-Smale sequences in the symmetric subspace $\mathcal{D}_a^{1,p}(\Omega, 0)^\phi$ are also Palais-Smale within the global space $\mathcal{D}^{1,p}_a(\Omega,0)$. The proof of this proposition utilizes a calculation from Clapp-Rios [14, Lemma A.1], together with a straightforward application of Minkowski's inequality for integrals to make the duality connection. We note that this observation allows for a more direct application of the known Struwe-type decomposition for weighted equations of this type.

Proposition 4.1. *Suppose $(u_\alpha) \subset \mathcal{D}^{1,p}_a(\Omega,0)^\phi$ is a Palais-Smale sequence for J in $\mathcal{D}^{1,p}_a(\Omega,0)^\phi$. That is, as $\alpha \to \infty$, $J(u_\alpha)$ converges to some real number and*

$$J'(u_\alpha) \to 0 \text{ in } \mathcal{D}^{-1,p'}_a(\Omega,0)^\phi.$$

Then, (u_α) is a Palais-Smale sequence for J in the non-symmetrized space $\mathcal{D}^{1,p}_a(\Omega,0)$. That is,

$$J'(u_\alpha) \to 0 \text{ in } \mathcal{D}^{-1,p'}_a(\Omega,0).$$

Moreover, if $u \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi$ solves (3.1), then $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\Omega,0)$.

Proof. Let $h \in C^\infty_c(\Omega)$ be an arbitrary test function. Let μ denote the normalized Haar measure on G, so that $\mu(G) = 1$, and define

$$\vartheta(x) = \int_G \phi(g)h(gx)\,\mathrm{d}\mu.$$

This rule clearly defines a function $\vartheta \in C^\infty_c(\Omega)^\phi$. Moreover, by Minkowski's inequality for integrals

$$\begin{aligned}\|\vartheta\|_{\mathcal{D}^{1,p}_a(\Omega,0)} &= \left(\int_\Omega \left|\int_G \phi(g)g^{-1}\nabla h(gx)\,\mathrm{d}\mu\right|^p |x|^{-ap}\,\mathrm{d}x\right)^{1/p} \\ &\le \int_G \left(\int_\Omega |\phi(g)g^{-1}\nabla h(gx)|^p\,|x|^{-ap}\,\mathrm{d}x\right)^{1/p}\mathrm{d}\mu.\end{aligned}$$

Then, simplifying, using that g, g^{-1} are linear isometries, and performing a change of variables, we obtain

$$\begin{aligned}\|\vartheta\|_{\mathcal{D}^{1,p}_a(\Omega,0)} &\le \int_G \left(\int_\Omega |\nabla h(gx)|^p\,|gx|^{-ap}\,\mathrm{d}x\right)^{1/p}\mathrm{d}\mu \\ &= \int_G \left(\int_\Omega |\nabla h(y)|^p\,|y|^{-ap}\,\mathrm{d}y\right)^{1/p}\mathrm{d}\mu \\ &= \|h\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \qquad (4.1)\end{aligned}$$

Next, as in Clapp-Rios [14, Lemma A.1], if $u \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi$ then $u(gx) = \phi(g)u(x)$ and $\phi(g)\nabla u(x) = g^{-1}\nabla u(gx)$ so

$$\begin{aligned}&\langle J'(u), \vartheta\rangle \\ &= \int_\Omega |\nabla u(x)|^{p-2}\,\nabla u(x)\cdot\nabla\vartheta(x)\,|x|^{-ap}\,\mathrm{d}x - \int_\Omega |u(x)|^{q-2}\,u(x)\vartheta(x)\,|x|^{-bq}\,\mathrm{d}x \\ &= \int_\Omega\int_G |\nabla u(x)|^{p-2}\,\phi(g)\nabla u(x)\cdot(g^{-1}\nabla h(gx))\,|x|^{-ap}\,\mathrm{d}\mu\mathrm{d}x \\ &\quad - \int_\Omega\int_G |u(x)|^{q-2}\,\phi(g)u(x)h(gx)\,|x|^{-bq}\,\mathrm{d}\mu\mathrm{d}x \\ &= \int_\Omega\int_G |\nabla u(gx)|^{p-2}\,(g^{-1}\nabla u(gx))\cdot(g^{-1}\nabla h(gx))\,|x|^{-ap}\,\mathrm{d}\mu\mathrm{d}x \\ &\quad - \int_\Omega\int_G |u(x)|^{q-2}\,u(gx)h(gx)\,|x|^{-bq}\,\mathrm{d}\mu\mathrm{d}x.\end{aligned}$$

Then, using that g^{-1} is a linear isometry and applying Fubini's Theorem, we get

$$\begin{aligned}
&\langle J'(u), \vartheta\rangle \\
&= \int_G \int_\Omega |\nabla u(gx)|^{p-2} \nabla u(gx) \cdot \nabla h(gx) \, |gx|^{-ap} \, \mathrm{d}x \mathrm{d}\mu \\
&\quad - \int_G \int_\Omega |u(gx)|^{q-2} u(gx) h(gx) \, |gx|^{-bq} \, \mathrm{d}x \mathrm{d}\mu \\
&= \int_G \int_\Omega |\nabla u(y)|^{p-2} \nabla u(y) \cdot \nabla h(y) \, |y|^{-ap} \, \mathrm{d}y \mathrm{d}\mu \\
&\quad - \int_G \int_\Omega |u(y)|^{q-2} u(y) h(y) \, |y|^{-bq} \, \mathrm{d}y \mathrm{d}\mu \\
&= \langle J'(u), h\rangle .
\end{aligned}$$

In particular,

$$\langle J'(u_\alpha), h\rangle = \langle J'(u_\alpha), \vartheta\rangle .$$

Consequently, if u solves (3.1), then $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$. Finally, combining this with (4.1) and recalling that $\vartheta \in C^\infty_c(\Omega)^\phi \subseteq \mathcal{D}^{1,p}_a(\Omega, 0)^\phi$, we see that

$$\begin{aligned}
|\langle J'(u_\alpha), h\rangle| = |\langle J'(u_\alpha), \vartheta\rangle| &\leq \|J'(u_\alpha)\|_{\mathcal{D}^{-1,p'}_a(\Omega,0)^\phi} \|\vartheta\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \\
&\leq \|J'(u_\alpha)\|_{\mathcal{D}^{-1,p'}_a(\Omega,0)^\phi} \|h\|_{\mathcal{D}^{1,p}_a(\Omega,0)} .
\end{aligned}$$

The assertion now follows by a density argument. □

5. Symmetric Global Compactness

We now establish a symmetric global compactness result for Palais-Smale sequences in the symmetrized space. Throughout this section, G denotes a closed subgroup of the orthogonal group $O(n)$ acting on a G-invariant bounded domain $\Omega \subset \mathbb{R}^n$, containing the origin, and $\phi : G \to \{\pm 1\}$ is a continuous homomorphism of groups. Furthermore, (G, ϕ) are assumed to satisfy (P1)-(P2).

Proposition 5.1. *Assume $a < b$. Suppose there exists a Palais-Smale sequence $(u_\alpha) \subset \mathcal{D}^{1,p}_a(\Omega, 0)^\phi$ for J such that, as $\alpha \to \infty$,*

$$J(u_\alpha) \to c^\phi, \quad J'(u_\alpha) \to 0 \text{ in } \mathcal{D}^{-1,p'}_a(\Omega, 0)^\phi.$$

Then, there exists a non-trivial solution to (3.2).

Proof. By an application of Proposition 4.1, the sequence (u_α) is Palais-Smale in $\mathcal{D}^{1,p}_a(\Omega, 0)$ with strictly positive energy level c^ϕ. The next step is to apply the Struwe-type decomposition in Chernysh [6, Theorem 1]. This yields $v_0 \in \mathcal{D}^{1,p}_a(\Omega, 0)$, an integer $k \geq 0$ (the case $k = 0$ is that of relative compactness) along with non-trivial functions $w_1, \dots, w_k \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ and sequences of radii $(\lambda^{(j)}_\alpha)$ such that

1. $u_\alpha \rightharpoonup v_0$ in $\mathcal{D}^{1,p}_a(\Omega, 0)$ and $J'(v_0) = 0$ on $\mathcal{D}^{1,p}_a(\Omega, 0)$,
2. for each index $j = 1, \dots, k$ the function $w_j \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ is a solution to (1.1) (i.e. $J'(w_j) = 0$ on $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ for every index $j = 1, \dots, k$,
3. the following energy decomposition holds: $c^\phi = J(v_0) + \sum_{j=1}^k J(w_j)$ with $J(v_0) \geq 0$ and $J(w_j) > 0$ for all admissible indices $j = 1, \dots, k$.

Moreover, as $\alpha \to \infty$,

$$u_\alpha - v_0 - \sum_{j=1}^k \left(\lambda^{(j)}_\alpha\right)^{-\gamma} w_j(\cdot/\lambda^{(j)}_\alpha) \to 0 \text{ in } \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0). \tag{5.1}$$

We now have two possible cases to consider.

(1) We first treat the case $v_0 \neq 0$. Owing to the fact that v_0 is the weak limit of the sequence (u_α) belonging to the closed subspace $\mathcal{D}^{1,p}_a(\Omega,0)^\phi$ of $\mathcal{D}^{1,p}_a(\Omega,0)$, it is automatic that $v_0 \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi$. As $v_0 \neq 0$ satisfies $J'(v_0) = 0$ in $\mathcal{D}^{1,p}_a(\Omega,0)^\phi$, we have $v_0 \in \mathcal{N}^\phi(\Omega)$ whence $J(v_0) \geq c^\phi$. Consequently, it follows from the energy expansion above that $k = 0$ and $c^\phi = J(v_0)$. After an extension by 0, we have $v_0 \in \mathcal{N}^\phi(\mathbb{R}^n)$ and so (3.2) has a non-trivial solution.

(2) Next we consider the case $v_0 = 0$ so that $J(v_0) = 0$. By the energy expansion, $k > 0$. However, since any admissible w_j belongs to $\mathcal{N}^\phi(\mathbb{R}^n)$ and necessarily satisfies $J(w_j) \geq c^\phi$, we must have $k = 1$. Examining Step 5 of Chernysh [6, Theorem 1] shows that this w_1 is, by construction, the weak limit of the rescaled sequence $x \mapsto \left(\lambda^{(1)}_\alpha\right)^\gamma u_\alpha(\lambda^{(1)}_\alpha x)$ and therefore $w_1 \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi$ is a solution to (3.2).

□

Proposition 5.2. *Assume $a = b \neq 0$ and (G,ϕ) is such that (P3) holds true. Suppose there exists a Palais-Smale sequence $(u_\alpha) \subset \mathcal{D}^{1,p}_a(\Omega,0)^\phi$ for J such that, as $\alpha \to \infty$,*

$$J(u_\alpha) \to c^\phi, \quad J'(u_\alpha) \to 0 \text{ in } \mathcal{D}^{-1,p'}_a(\Omega,0)^\phi.$$

Then, there exists non-trivial solution to (3.2).

Proof. As before, Proposition 4.1 implies that the given (PS)-sequence is also Palais-Smale when considered within the larger space $\mathcal{D}^{1,p}_a(\Omega,0)$. As in the proof of Proposition 5.1, we proceed one more by applying a Struwe-type decomposition (Chernysh [6, Theorem 2]). The only difference here is that we invoke property (P3) to reduce the bubble-tree to that we encountered in Proposition 5.1.

By [6, Lemma 6.1], after passing to a subsequence, there exists a weak limit $v_0 \in \mathcal{D}^{1,p}_a(\Omega,0)$ such that $u_\alpha \rightharpoonup v_0$ in $\mathcal{D}^{1,p}_a(\Omega,0)$ and pointwise a.e. on Ω, as $\alpha \to \infty$. Moreover,

(1) $\nabla u \to \nabla v_0$ pointwise a.e. on Ω,
(2) $J'(v_0) = 0$ on $\mathcal{D}^{1,p}_a(\Omega,0)$,
(3) $J(u_\alpha - v_0) \to c^\phi - J(v_0)$,
(4) $J'(u_\alpha - v_0) \to 0$ in $\mathcal{D}^{-1,p'}_a(\Omega,0)$.

Of course, being the weak limit of a sequence in $\mathcal{D}^{1,p}_a(\Omega,0)^\phi$, we must have that v_0 is ϕ-equivariant, i.e. $v_0 \in \mathcal{D}^{1,p}_a(\Omega,0)^\phi$. We now distinguish two possible cases. First, if $v_0 \neq 0$, then we proceed exactly as in the first part of Proposition 5.1. If instead $v_0 = 0$, the sequence (u_α) has a failure of compactness. Then, as in [6, Equation 8.1], we select sequences (y_α) in $\overline{\Omega}$ and (λ_α) in $(0,\infty)$ such that

$$\sup_{y\in\overline{\Omega}} \int_{B(y,\lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x = \int_{B(y_\alpha,\lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x = \delta$$

where $\delta > 0$ is chosen sufficiently small. By leveraging (P3), we can show that (up to a subsequence)

$$\frac{y_\alpha}{\lambda_\alpha} \to x_0 \tag{5.2}$$

for some $x_0 \in \mathbb{R}^n$. Indeed, should this sequence be unbounded, we may pass to a subsequence in such a way that

$$\frac{|y_\alpha|}{\lambda_\alpha} \to \infty, \quad |y_\alpha| > 4\lambda_\alpha.$$

Then, each $y_\alpha \neq 0$ so $y_\alpha/|y_\alpha|$ is a sequence on the unit sphere. Passing to yet another subsequence, we may assume that

$$\frac{y_\alpha}{|y_\alpha|} \to y, \quad \text{as } \alpha \to \infty.$$

Appealing to condition (P3) and borrowing ideas from Clapp-Rios [14], given any $m \in \mathbb{N}$, we may select elements $g_1, \dots, g_m \in G$ such that $g_i y \neq g_j y$, for all $i \neq j$. Consequently, there exists $\varepsilon > 0$ so that

$$|g_i y - g_j y| \geq 2\varepsilon > 0, \quad \forall i \neq j.$$

Using then that $\frac{y_\alpha}{|y_\alpha|} \to y$, it follows from this last identity that

$$\left| g_i \frac{y_\alpha}{|y_\alpha|} - g_j \frac{y_\alpha}{|y_\alpha|} \right| \geq \varepsilon \tag{5.3}$$

for all $\alpha \in \mathbb{N}$ sufficiently large. Hence, for all such α,

$$\frac{|g_i y_\alpha - g_j y_\alpha|}{\lambda_\alpha} \geq \varepsilon \frac{|y_\alpha|}{\lambda_\alpha}. \tag{5.4}$$

Because the right hand side tends to $+\infty$, we infer that

$$B(g_i y_\alpha, \lambda_\alpha) \cap B(g_j y_\alpha, \lambda_\alpha) = \varnothing,$$

for all $i \neq j$ and α large. But,

$$\begin{aligned}
\delta &= \int_{B(y_\alpha,\lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x \\
&= \int_{B(g_i y_\alpha,\lambda_\alpha)} \left|u_\alpha(g_i^{-1} z)\right|^q \left|g_i^{-1} z\right|^{-bq} \mathrm{d}z, \qquad\qquad (z := g_i x) \\
&= \int_{B(g_i y_\alpha,\lambda_\alpha)} \left|\phi(g_i^{-1}) u_\alpha(z)\right|^q |z|^{-bq} \, \mathrm{d}z \\
&= \int_{B(g_i y_\alpha,\lambda_\alpha)} |u_\alpha(z)|^q \, |z|^{-bq} \, \mathrm{d}z
\end{aligned}$$

It thereby follows that, for all α sufficiently large,

$$\int_{\mathbb{R}^n} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x \geq \sum_{i=1}^m \int_{B(g_i y_\alpha,\lambda_\alpha)} |u_\alpha(z)|^q \, |z|^{-bq} \, \mathrm{d}z = m\delta,$$

which contradicts the fact that (u_α) is a bounded sequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$. In short: we have proven that y_α/λ_α is bounded and thus, passing to a subsequence, we may assume (5.2). It follows from [6, Theorem 2] that the bubble-tree decomposition of the sequence (u_α) is of the form (5.1). From here, we complete the argument by following the second part of the proof from Proposition 5.1. □

We are now ready to supply the proof of our main result.

Proof of Theorem 1.1. Let $n \geq 4$ and put $k = k(n) := \left\lfloor \frac{n}{2} \right\rfloor - 1$. For each $\alpha \geq 0$ and every tuple $\mathfrak{m} = (m_1, \ldots, m_k)$ such that $(\alpha, \mathfrak{m})$ satisfies condition (1.4), the construction detailed in §2 yields a group and homomorphism pair $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,\mathfrak{m}})$ satisfying properties (P1)-(P2) by virtue of Lemma 2.3.

If $a < b$ in the problem parameters of (1.1), an application of Corollary 3.4 together with Proposition 5.1, for Ω equal to the unit ball in $\mathbb{R}^n$, produces, for any such pair $(\alpha, \mathfrak{m})$, a non-trivial solution $u_{\alpha,\mathfrak{m}}$ to problem (3.2) which is $(\alpha, \mathfrak{m})$-symmetric. Since $\phi_{\alpha,\mathfrak{m}}$ is surjective and $u_{\alpha,\mathfrak{m}}$ is non-trivial, it is automatic that $u_{\alpha,\mathfrak{m}}$ is sign-changing. Then, Proposition 4.1 applied with $\Omega = \mathbb{R}^n$ ensures that $u_{\alpha,\mathfrak{m}}$ solves the unrestricted problem (1.1).

If we are instead in the case $a = b \neq 0$, and the pair $(\alpha, \mathfrak{m})$ satisfies the additional condition (1.5), the argument proceeds exactly as in the case $a < b$, but replaces Proposition 5.1 with Proposition 5.2.

Finally, we treat the case $a = b = 0$. Although condition (P3) will not hold unless (1.5) is satisfied, it is easy to see from the definition of $G_{\alpha,\mathfrak{m}}$ and (2.5) that the weaker condition

$$\dim\left(\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x)\right) > 0 \quad \text{or} \quad \mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x) = \{x\}$$

is valid for all $x \in \mathbb{R}^n$. Consequently, an application of Clapp-Rios [14, Theorem 3.1] with Ω equal to the unit ball ensures the existence of a non-trivial solution $u_{\alpha,\mathfrak{m}}$ of the problem (1.3) that is $(\alpha, \mathfrak{m})$-symmetric. Furthermore, being that $\phi_{\alpha,\mathfrak{m}}$ is surjective by construction, each $u_{\alpha,\mathfrak{m}}$ is sign-changing.

The second part of the theorem follows by combining Propositions 2.7-2.8. □

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