A GLOBAL COMPACTNESS THEORY FOR CAFFARELLI-KOHN-NIRENBERG PROBLEMS WITH APPLICATIONS

Edward Chernysh

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

Under the supervision of Professor Jérôme Vétois and Professor Pengfei Guan.

Department of Mathematics and Statistics,
McGill University,
Montréal, Québec,
April 2026.

Abstract

In this thesis, we investigate critical Caffarelli-Kohn-Nirenberg problems in $\mathbb{R}^n$, for dimensions $n \geq 2$. That is, we study the problem

$$-\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u\right) = |x|^{-bq} |u|^{q-2} u \quad \text{in } \Omega, \tag{A}$$

where $1 < p < n$, $a < (n-p)/p$, $a \leq b < a+1$, q is the critical Caffarelli-Kohn-Nirenberg exponent, and $\Omega \subseteq \mathbb{R}^n$ is a domain containing the origin. More precisely, we first establish a global compactness theory (also called a Struwe-type decomposition) extending the result obtained by Mercuri-Willem [52] for the unweighted p-Laplacian (and originally by Struwe [63] for the unweighted Laplacian) to the weighted Caffarelli-Kohn-Nirenberg regime, for the full range of admissible parameters. We show that, depending on the parameter ranges, the resulting bubble-tree decomposition falls into three distinctive categories. First, if $a < b$, then any loss of compactness must occur at the origin, exclusively. Especially, we obtain a decomposition result which holds even in domains with non-smooth boundary. In sharp contrast to both $a = b = 0$ and $a < b$, when $a = b \neq 0$, we discover a unique bubbling configuration involving three potential limiting problems. Of important note here is that these limiting problems include both weighted and unweighted equations. In particular, for both $a < b$ and $a = b \neq 0$, we give new adapted rescaling laws to overcome the loss of translation invariance due to the presence of weights.

Finally, by leveraging this global compactness theorem, we establish the existence of entire sign-changing solutions to the Caffarelli-Kohn-Nirenberg equation, posed in $\mathbb{R}^n$ for dimensions $n \geq 4$. In fact, we demonstrate that for an infinite number of mutually exclusive symmetry-types, there exists an entire sign-changing solution to the Caffarelli-Kohn-Nirenberg problem respecting the aforementioned symmetry-type. In particular, we give what we believe is the first result establishing the existence of entire nodal solutions in the full parameter range, and especially, the range $a < 0$.

Abrégé

Dans cette thèse, nous étudions des problèmes critiques de Caffarelli-Kohn-Nirenberg dans $\mathbb{R}^n$, pour les dimensions $n \geq 2$. Autrement dit, nous étudions le problème (A) où Ω est un domaine contenant l'origine, $1 < p < n$, $a < (n-p)/p$, $a \leq b < a+1$ et q est l'exposant critique de Caffarelli-Kohn-Nirenberg. Plus précisément, nous établissons d'abord une théorie de la compacité globale (également appelée décomposition de type Struwe) généralisant le résultat obtenu par Mercuri-Willem [52] pour le p-laplacien non pondéré (et initialement par Struwe [63] pour le laplacien non pondéré) au régime de Caffarelli-Kohn-Nirenberg pondéré, pour l'ensemble des valeurs admissibles des paramètres. Nous montrons que, selon les plages de valeurs des paramètres, la décomposition en arbre à bulles résultante se divise en trois catégories distinctes. Premièrement, si $a < b$, alors toute perte de compacité se produit exclusivement à l'origine. En particulier, nous obtenons un résultat de décomposition valable même dans les domaines à bord non lisse. Contrairement aux cas $a = b = 0$ et $a < b$, lorsque $a = b \neq 0$, nous découvrons une configuration de bulles unique impliquant trois problèmes limites potentiels. Il est important de noter que ces problèmes limites incluent des équations pondérées et non pondérées. En particulier, pour $a < b$ et $a = b \neq 0$, nous proposons de nouvelles lois de rééchelonnement adaptées afin de compenser la perte d'invariance par translation due à la présence de poids.

Enfin, en nous appuyant sur ce théorème de compacité globale, nous établissons l'existence de solutions changeant de signe entières de l'équation de Caffarelli-Kohn-Nirenberg, posée dans $\mathbb{R}^n$ pour les dimensions $n \geq 4$. En fait, nous démontrons que pour un nombre infini de types de symétrie mutuellement exclusifs, il existe une solution changeant de signe entière du problème de Caffarelli-Kohn-Nirenberg respectant le type de symétrie susmentionné. En particulier, nous présentons ce que nous pensons être le premier résultat établissant l'existence de solutions nodales entières pour l'ensemble des valeurs des paramètres, et notamment, dans le cas $a < 0$.

Acknowledgments

First and foremost, I would like to express my deepest gratitude toward my supervisors, Professors Jérôme Vétois and Pengfei Guan, for their unwavering support and guidance throughout both my master's and doctoral studies. I could not have asked for better guidance and more dedicated supervisors.

I am especially indebted to Jérôme; this work would not have been possible without his constant availability, his patience, and the many hours he spent in discussion with me. I also want to give a special thanks to Pengfei for his invaluable perspective and for the support he provided throughout this journey; his high standards and expertise in geometric analysis have deeply influenced this work.

I am very grateful to the Department of Mathematics and Statistics here at McGill, for their hospitality and for providing a stimulating environment for research. This work was made possible by the financial support of the Natural Sciences and Engineering Research Council of Canada (NSERC) through a CGS-D award, as well as the Institut des sciences mathématiques (ISM). I also wish to thank the Department and my supervisors for the additional funding and resources provided, which allowed me to focus entirely on this research.

To my wife: thank you for being my rock. Thank you for always being there, for your boundless patience, and for tolerating my endless mathematical ramblings.

Finally, I want to wholeheartedly thank my parents. You have supported me in every way imaginable from the very beginning. Thank you for your love, your encouragement, and for your immense grace in tolerating my long absences.

Remerciements

Avant toute chose, je tiens à exprimer ma profonde gratitude envers mes directeurs de recherche, les professeurs Jérôme Vétois et Pengfei Guan, pour leur soutien inestimable et leurs précieux conseils tout au long de mes études de maîtrise et de doctorat. Je n'aurais pu espérer un meilleur encadrement ni des directeurs de recherche plus dévoués.

Je suis particulièrement reconnaissant envers Jérôme; ce travail n'aurait pas été possible sans sa disponibilité constante, sa patience et les nombreuses heures qu'il a consacrées en discussion avec moi. Je tiens également à remercier Pengfei pour son point de vue inestimable et pour le soutien qu'il m'a apporté tout au long de ce parcours ; son exigence et son expertise en analyse géométrique ont profondément influencé ce travail.

Je suis très reconnaissant au Département de mathématiques et de statistique de l'Université McGill pour son accueil et pour l'environnement stimulant qu'il offre pour la recherche et les études. Ce travail a été rendu possible grâce au soutien financier du Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG) par l'entremise d'une bourse d'études supérieures du Canada (BEC) D, ainsi que de l'Institut des sciences mathématiques (ISM). Je tiens aussi à remercier le Département et mes directeurs de recherche pour le financement et les ressources supplémentaires qu'ils m'ont fournis et qui m'ont permis de me consacrer pleinement à cette recherche.

À mon épouse : merci d'être toujours là, pour ta patience infinie et pour supporter mes interminables digressions mathématiques.

Enfin, je tiens à remercier du fond du cœur mes parents. Vous m'avez supporté de toutes les façons possibles depuis le tout début. Merci pour votre amour, vos encouragements et votre immense compréhension face à mes longues absences.

Declaration of Authorship

The author certifies that the contents of this thesis are original scholarship, unless stated otherwise explicitly. The main results and proofs contained within Chapter 2 are entirely the independent works of the author, and consists of the results presented in the paper Chernysh [14]. For the sake of transparency, we mention that Theorem 1.1.1 (applying to $a < b$) was first established by the author during his MSc at McGill University (see Chernysh [12]). However, we include the statement, and proof, of said result in this manuscript both to increase the quality of exposition, and also because the proof is essential within the more difficult case $a = b \neq 0$. We refer the reader to discussions in §1.1 and to Chapter 2 for further comments on this. Meanwhile, we mention that the contents of Chapter 3 consist of the research found within the paper Chernysh [13].

Contents

Chapter 1

Background and Introduction

In this thesis, we address and investigate questions related to the analysis of critical quasilinear equations with weights in $\mathbb{R}^n$. More precisely, we are concerned with the analysis and characterization of the possible lack of compactness, for approximating sequences (Palais-Smale), associated to these equations, as well as the questions of existence and multiplicity of entire solutions (i.e. solutions to the Caffarelli-Kohn-Nirenberg problem posed in all of $\mathbb{R}^n$) which are nodal (i.e. solutions which change sign).

To be explicit, for dimensions $n \geq 2$, we study the critical Caffarelli-Kohn-Nirenberg equation

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = |x|^{-bq}\,|u|^{q-2}\,u \quad \text{in } \Omega, \\ u \in \mathcal{D}_a^{1,p}(\Omega, 0) \end{cases} \tag{A}$$

where $\Omega \subseteq \mathbb{R}^n$ is a domain containing the origin. The parameters a, b, p in problem (A) are subject to the constraints

$$1 < p < n, \quad a < \frac{n-p}{p}, \quad \text{and } a \leq b < a+1. \tag{1.0.1}$$

Here, q denotes the critical Caffarelli-Kohn-Nirenberg (often abbreviated CKN) exponent, given below by

$$q := \frac{np}{n - p(1 + a - b)}. \tag{1.0.2}$$

We observe that, when $a = b$, one has $q = p^*$, with $p^* = np/(n-p)$ denoting the usual critical Sobolev exponent. Finally, the ambient energy space $\mathcal{D}_a^{1,p}(\Omega, 0)$ is defined to be the completion of those smooth functions with compact support $\Omega \to \mathbb{R}$, the space of which is written $C_c^\infty(\Omega)$, under the *weighted homogeneous Sobolev* norm

$$\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|\nabla u\|_{L^p(\Omega,|x|^{-ap})} = \left(\int_\Omega |\nabla u(x)|^p \, |x|^{-ap} \, \mathrm{d}x\right)^{1/p}.$$

We shall discuss this specialized Sobolev space in greater detail, particularly within Chapter 2, where we recall essential properties the space $\mathcal{D}_a^{1,p}(\Omega, 0)$ satisfies (e.g. Rellich-Kondrachov and Riesz-representation theorems). Said properties will be essential and liberally invoked within Chapter 2, and are required in order to accurately characterize the possible loss of compactness one may encounter regarding Palais-Smale sequences associated to the problem (A).

We should also explicitly acknowledge that the problem (A) is the Euler-Lagrange equation associated to the Caffarelli-Kohn-Nirenberg inequality in a domain $\Omega \subseteq \mathbb{R}^n$, which asserts that there exists a constant $C > 0$ such that

$$\left(\int_\Omega |u(x)|^q \, |x|^{-bq} \, \mathrm{d}x\right)^{1/q} \leq C \left(\int_\Omega |\nabla u(x)|^p \, |x|^{-ap} \, \mathrm{d}x\right)^{1/p} \tag{CKN}$$

for all $u \in C_c^\infty(\Omega)$ (see the famous paper of Caffarelli-Kohn-Nirenberg [9]).

For purposes of motivation, we should also take a moment to point out that both sides of the Caffarelli-Kohn-Nirenberg inequality, and consequently it's associated Euler-Lagrange problem (A), are naturally invariant under an appropriate rescaling. That is, defining the homogeneity exponent

$$\gamma := \frac{n - bq}{q} = \frac{n - p(1+a)}{p}, \tag{1.0.3}$$

and taking arbitrary $\lambda > 0$, we have that $v(x) := \lambda^\gamma u(\lambda x)$ satisfies both

$$\left(\int_{\Omega/\lambda} |v(x)|^q \, |x|^{-bq} \, \mathrm{d}x\right)^{1/q} = \left(\int_\Omega |u(x)|^q \, |x|^{-bq} \, \mathrm{d}x\right)^{1/q}$$

and

$$\left(\int_{\Omega/\lambda} |\nabla v(x)|^p \, |x|^{-ap} \, \mathrm{d}x\right)^{1/p} = \left(\int_{\Omega} |\nabla u(x)|^p \, |x|^{-ap} \, \mathrm{d}x\right)^{1/p}.$$

Moreover, if u is a solution to (A), then v solves the same problem, but posed in the rescaled domain Ω/λ.

In addition to being the Euler-Lagrange problem arising from an important inequality, the weighted problem (A) arises as a natural generalization of the critical p-Laplace problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2} \, u \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{1.0.4}$$

Here, Δ_p denotes the p-Laplacian, which is given by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \, \nabla u)$$

when applied to a function u, and $W_0^{1,p}(\Omega)$ is the usual unweighted homogeneous Sobolev space for the p-Laplacian. We remark that this purely critical p-Laplace problem (1.0.4) has received considerable attention in recent years, particularly with regards to questions of existence and multiplicity (notably Clapp-Rios [24] and Clapp-Vicente-Benítez [21]), in addition to global compactness theorems (e.g. Mercuri-Willem [52]). Furthermore, the positive solutions to (1.0.4) in $\Omega = \mathbb{R}^n$ have been completely classified; as the history of this classification is lengthy and delicate, we refer the reader to the end of this section for the precise details.

In the special case of the unweighted critical Laplace problem (i.e. $a = b = 0$ with $p = 2$), the problem (1.0.4) reduces to

$$\begin{cases} -\Delta u = |u|^{2^*-2} \, u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.0.5}$$

In taking $\Omega = \mathbb{R}^n$, and replacing $H_0^1(\mathbb{R}^n)$ with $\mathcal{D}^{1,2}(\mathbb{R}^n)$, this becomes the Yamabe problem, for which there is an especially rich body of literature. We shall return to this later, when we discuss nodal solutions.

The organization of this thesis is as follows. For what remains of this introduction we elaborate in greater detail upon the important literature related to (A) and the main results of this thesis, while providing the necessary variational framework to communicate this history and the flagship theorems of this manuscript. Concretely, in the next section of this chapter, we give an overview of Struwe-type decompositions, with particular emphasis on known results for the unweighted problems (1.0.4)-(1.0.5), and present our first two main results, which constitutes a general Struwe-type decomposition for the weighted problem (A) in the full range of parameters (1.0.1). In addition to an informal discussion of these results, we will provide an overview of important theorems in the literature, relating to the conclusions of our two theorems. In the subsequent section (i.e. §1.2), we present our final main theorem, which instead addresses the question of existence and multiplicity of sign-changing solutions to (A) for the whole space $\Omega = \mathbb{R}^n$. Of course, we shall also describe important related results in the literature, particularly how our nodal existence theorem improves conclusions drawn in the unweighted case.

1.1 Global Compactness

As mentioned earlier, our first two main results are generalizations of the Struwe-type decomposition for the unweighted problem (1.0.4). Decomposition results of this type, also known as *global compactness theorems*, completely describe the possible loss of compactness for Palais-Smale sequences associated to the energy functional describing a partial differential equation. In the context of the Caffarelli-Kohn-Nirenberg problem (A), the corresponding energy functional is given below explicitly by

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u(x)|^p\,|x|^{-ap}\,\mathrm{d}x - \frac{1}{q}\int_\Omega |u(x)|^q\,|x|^{-bq}\,\mathrm{d}x \tag{1.1.1}$$

for $u \in \mathcal{D}^{1,p}_a(\Omega, 0)$. We reiterate for the sake of exposition that $\mathcal{D}^{1,p}_a(\Omega, 0)$ is the weighted homogeneous Sobolev space obtained by the completion of $C_c^\infty(\Omega)$ with respect to the weighted norm

$$\|u\|_{\mathcal{D}^{1,p}_a(\Omega,0)} := \left(\int_\Omega |\nabla u(x)|^p\,|x|^{-ap}\,\mathrm{d}x\right)^{1/p}.$$

A function $u \in \mathcal{D}^{1,p}_a(\Omega, 0)$ is said to be a (weak) solution to (A) provided the (Fréchet) derivative of J vanishes at u, i.e. if $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$, where

$$\langle J'(u), h\rangle = \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla h) |x|^{-ap} \, \mathrm{d}x - \int_\Omega |u|^{q-2} uh |x|^{-bq} \, \mathrm{d}x.$$

In the above, we write $(\cdot, *)$ to represent the usual dot-product on $\mathbb{R}^n$, while the symbol $\langle\cdot, *\rangle$ denotes the duality pairing. We interpret the quantity $J(u)$ as the energy of a function in $\mathcal{D}^{1,p}_a(\Omega, 0)$, and the Sobolev space $\mathcal{D}^{1,p}_a(\Omega, 0)$ to be the vector space consisting of all "suitable" functions whose energy (with respect to this energy functional J) is finite.

Since solutions of (A) are simply the critical points of the energy functional J, it is natural to consider sequences of functions which naively approach a critical point. That is, in the hopes of exhibiting a non-trivial solution to (A), one might attempt to build a sequence of functions (u_α) belonging to the energy space $\mathcal{D}^{1,p}_a(\Omega, 0)$ for which $J'(u_\alpha) \to 0$, with respect to the strong operator topology, as $\alpha \to \infty$. A particularly useful class of such sequences are the *Palais-Smale sequences* (often abbreviated by(PS)-sequences), which we define below. These are effectively the "approximating sequences" described loosely above, which have uniformly bounded energy and which seek to approximate a critical point of J.

Definition 1.1.1. Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$ and E be a C^1-functional defined on X. A sequence (x_α) in X is said to be Palais-Smale, with respect to E, provided

1. $E(x_\alpha)$ is bounded independently of $\alpha \in \mathbb{N}$,

2. $E'(x_\alpha) \to 0$ strongly in the topological dual space E^*, as $\alpha \to \infty$.

Should the limit $\lim_{\alpha\to\infty} E(x_\alpha)$ exist, then this quantity is called the *energy level* of the (PS)-sequence (x_α). In the case that E is the energy functional corresponding to a partial differential equation, we will allow ourselves to say that a sequence is Palais-Smale with respect to the associated partial differential equation, provided it is a (PS)-sequence for the functional E. We remark that the functional E is said to satisfy the *Palais-Smale condition* if every Palais-Smale sequence has a subsequence converging strongly in X. More generally, given an energy level c, we say that E satisfies the Palais-Smale condition at energy level c (often written as the $(\mathrm{PS})_c$-condition) if every Palais-Smale sequence at energy level c has a subsequence converging strongly in X.

Although it is often possible to construct non-trivial Palais-Smale sequences for the energy functional J, such a sequence can be tricky to work with concretely. Certainly, given a Palais-Smale sequence (u_α) in $\mathcal{D}^{1,p}_a(\Omega, 0)$ for J with non-zero energy level, one might hope to utilize a suitable compact embedding theorem (or verify the Palais-Smale condition) in order to ensure the existence of a subsequence (u_β) converging strongly, in an appropriate sense, to a function u and proving this u is a solution to the problem (A). Unfortunately, guaranteeing the existence of a subsequence (for a general (PS)-sequence) converging strongly in $\mathcal{D}^{1,p}_a(\Omega, 0)$, is impossible. Indeed, this difficulty is inherent to critical equations, as there is a non-linearity involving the unknown function u for which the Sobolev embeddings are continuous, but not compact. Nevertheless, there is much to be gained from an understanding of the possible lack of compactness such a sequence can possess. As a matter of fact, accurate descriptions of compactness failures have a lengthy history, and a proven track record regarding mathematical utility. For example, important works of Lions [48]-[49] (and other authors, for other operators and equations) describe the possible limit profiles of a Palais-Smale sequence at the level of measures, in terms of weak and weak* convergence. Such results can sometimes be used to demonstrate existence of solutions, but are not always sufficient for these purposes. A powerful observation, first made by Struwe [63] in 1984, was that, in the limit, a Palais-Smale sequence for the unweighted critical Laplace problem (1.0.5) (i.e. a (PS)-sequence for J with $a = b = 0$ and $p = 2$), posed in a sufficiently well-behaved domain, decomposes into a solution of (1.0.5) perturbed by finitely many translated and rescaled solutions to the same problem, but in $\mathbb{R}^n$. Moreover, this decomposition takes place in both senses with respect to the strong norm topology and the energy functional J. This result was the first global compactness theorem, and as such, these results are often respectfully called *Struwe-type decompositions.*

In other words, although a general (PS)-sequence for the unweighted critical problem (1.0.5) may not have a subsequence converging strongly in the energy space, there *does* exist a subsequence that (in the limit) splits into a solution to the problem itself, perturbed by a finite sum of translated rescalings of solutions to a limiting problem in $\mathbb{R}^n$. Furthermore, the energy level is *conserved*, in the sense that the energy of the (PS)-sequence is precisely the sum of the respective energies of all solutions. These rescaled translations which perturb the solution in Ω, are frequently called "bubbles". Loosely speaking, Struwe's theorem provides a complete asymptotic expansion of the Palais-Smale sequence in the energy space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Struwe's groundbreaking theorem has found numerous applications in questions of existence; we mention in particular the famous Bahri-Coron result [4], which highlights the so called "effect of topology" in the domain (e.g. a sufficiently thick annulus).

Of further relevance to our results, is the well known extension of Stuwe's decomposition to the p-Laplacian operator, established by Mercuri-Willem [52] in 2012. Within this paper, the authors demonstrate that a Palais-Smale sequence for problem (1.0.4) whose negative part decays in $L^{p^*}(\Omega)$, posed in a sufficiently smooth domain, possesses a subsequence which obeys the Struwe decomposition, i.e. it splits asymptotically into a solution of the problem (1.0.4) perturbed by finitely many rescaled translations of *positive* solutions to the same problem, but posed in the entirety of $\mathbb{R}^n$. We also refer the reader to the work of Alves [2] for related results for a similar problem involving the p-Laplacian.

Decomposition results of this type have been used to shed light on both the existence and multiplicity of solutions, see for instance the important works of [30, 22, 67, 5, 24]. In addition, we note that decompositions in spirit of Struwe's have been established in other contexts. Examples include Hebey-Robert [42] for Paneitz type operators, Saintier [58] for p-Laplace operators on compact manifolds, El-Hamidi-Vétois [33] for anisotropic operators, Almaraz [1] for non-linear boundary conditions and Mazumdar [50] for polyharmonic operators on compact manifolds.

As an aside, we also emphasize that understanding the limit profile of a Palais-Smale sequence itself provides valuable information on the partial differential equation. Indeed, should the Palais-Smale sequence be non-negative, in the sense that every $u_\alpha \geq 0$, (or even if the negative part of u_α decays in the appropriate sense) then any possible bubble appearing must be a positive solution to

$$\begin{cases} -\Delta u = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \end{cases} \tag{1.1.2}$$

the positive solutions to which have been completely classified (we refer the reader to this end of this section for further details regarding this classification).

Having now provided an informal high level discussion of Struwe-type decompositions, along with an imprecise treatment of the convergences it guarantees, we are now in a position to state our first main results of this manuscript. The Struwe-type decomposition for the Caffarelli-Kohn-Nirenberg problem (A) exhibits distinctive phenomena in the cases $a < b$ and $a = b$, and as such, for the clarity of our exposition, we state these separately as two independent theorems. In addition, it will become readily apparent from the proofs that the case $a = b \neq 0$ is the most challenging, and features a new blow-up configuration, which occurs only in this case. This new bubbling configuration is a result of the loss in translation invariance due to the weights and, in order to account for energy loss in this regime, we develop a new rescaling law (which we shall denote $\tau^-_{y,\lambda}$) capable of exhaustively capturing this concentration phenomenon. We shall discuss this new rescaling law in greater detail shortly, after having stated our Struwe-type decomposition. Regarding our main results, we shall begin by presenting the decomposition for $a < b$, which nonetheless exhibits deviations from the unweighted model case.

Theorem 1.1.1 (Chernysh [12])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin and assume (1.0.1). Let (u_α) be a Palais-Smale sequence for (1.1.1) for which $\lim_{\alpha\to\infty} J(u_\alpha)$ exists. Suppose $a < b$ and let $\gamma > 0$ be the homogeneity exponent from (1.0.3). Then, there exists a subsequence (u_β) of (u_α) along with*

(1) a solution v_0 of (A);

(2) finitely many non-trivial solutions $w_1, \dots, w_k$ for the weighted limiting problem

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u\right) = |x|^{-bq} |u|^{q-2} u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0) \end{cases} \tag{B}$$

and, for index each $j = 1, \dots, k$, a corresponding sequence $(\lambda_\beta^{(j)})$ in $(0, \infty)$ such that

$$\lambda_\beta^{(j)} \to 0, \quad \text{as } \beta \to \infty.$$

Furthermore, as $\beta \to \infty$, there hold the following.

(i) Decomposition in the Energy Space:

$$u_\beta - v_0 - \sum_{j=1}^{k} \left(\lambda_\beta^{(j)}\right)^{-\gamma} w_j \left(\frac{\cdot}{\lambda_\beta^{(j)}}\right) \to 0 \quad \text{in } \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0) \tag{1.1.3}$$

(ii) Norm Decomposition:

$$\|u_\beta\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \to \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} + \sum_{j=1}^{k} \|w_j\|^p_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}$$

(iii) Energy Decomposition:

$$J(u_\beta) \to J(v_0) + \sum_{j=1}^{k} J_{0,\infty}(w_j).$$

In the statement above, we write $J_{0,\infty}$ to denote the energy functional corresponding to the weighted limiting problem arising in (B), i.e. for $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ we have

$$J_{0,\infty}(u) := \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u(x)|^p \, |x|^{-ap} \, \mathrm{d}x - \frac{1}{q} \int_{\mathbb{R}^n} |u(x)|^q \, |x|^{-bq} \, \mathrm{d}x. \tag{1.1.4}$$

Of course, a (weak) solution to (B) is a function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ for which there holds $\left\langle J'_{0,\infty}(u), h \right\rangle = 0$ for all $h \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$, where

$$\left\langle J'_{0,\infty}(u), h \right\rangle = \int_{\mathbb{R}^n} |\nabla u|^{p-2} \, (\nabla u, \nabla h) \, |x|^{-ap} \, \mathrm{d}x - \int_{\mathbb{R}^n} |u|^{q-2} \, uh \, |x|^{-bq} \, \mathrm{d}x.$$

We point out that, exactly as in the unweighted model case, whenever our (PS)-sequence (u_α) is non-negative, the resulting functions v_0, w_j are themselves non-negative. For certain parameter ranges (see below), these positive w_j's are completely classified and their respective energies precisely understood.

In addition, we observe that in the conclusions of Theorem 1.1.1, every one of the "bubbles" (i.e. the rescaled w_j's appearing in (1.1.3)) necessarily concentrate at the origin. This stands in sharp contrast to the model case $a = b = 0$, considered by Struwe [63] for $p = 2$ and later by Mercuri-Willem [52] for general $1 < p < n$, wherein the bubbles may concentrate at any point in Ω. In other words, if there is any loss of compactness, it must take place at the origin. Being that the origin is an interior point of the domain Ω, this loss of compactness is far from the boundary, and as such, we obtain a decomposition result that does not require any kind of boundary regularity. In hindsight, this origin enforced bubbling is natural, owing to the fact that in the $a < b$ regime, the nonlinearity $q > p$ is strictly smaller than the critical Sobolev exponent p^*, and hence the natural embedding $\mathcal{D}^{1,p}_a(\Omega, 0) \hookrightarrow L^q_{\text{loc}}(\Omega \setminus \{0\})$ is compact.

Next we turn our attention to the balanced setting where $a = b \neq 0$ within the parameters (1.0.1). In this regime, we have $q = p^*$ and consequently, in addition to the complexities imposed by the weight at the origin, we have the critical nonlinearity on u to appease at all points of the domain Ω. As a result, the case $a = b \neq 0$ exhibits a new bubbling configuration, requiring an energy space decomposition involving three distinct limiting problems. This new bubbling configuration presents a strong deviation from the model unweighted case (1.0.4), is unique to the $a = b \neq 0$ regime, and combines the difficulties introduced by both the weights and a critical exponent. In fact, if we have concentration away from the origin (or even slowly at the origin), the resulting bubble will not "feel" the weight, and satisfies an unweighted critical equation, either in $\mathbb{R}^n$ or a half-space. Formally, the unique bubbling configuration arising in the $a = b \neq 0$ case involves three limiting problems, belonging to both critical weighted and unweighted categories. First, we have the weighted problem (B) that we previously encountered within the $a < b$ case. The second, is an entire unweighted analogue given by

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}\, u \quad \text{in } \mathbb{R}^n \\ u \in \mathcal{D}^{1,p}\left(\mathbb{R}^n\right), \end{cases} \tag{C}$$

whose associated energy functional is given by

$$J_\infty : \mathcal{D}^{1,p}(\mathbb{R}^n) \supseteq \mathcal{D}^{1,p}(\mathbb{H}) \to \mathbb{R},$$
$$J_\infty(u) := \int_{\mathbb{R}^n} |\nabla u(x)|^p \,\mathrm{d}x - \int_{\mathbb{R}^n} |u(x)|^{p^*} \,\mathrm{d}x.$$

Furthermore, as one cannot generally rule out bubbling near or on the boundary of Ω, we are forced to also consider the unweighted limiting problem set in the half-space $\mathbb{H} := \{x \in \mathbb{R}^n : x_n > 0\}$,

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = |u|^{p^*-2}\,u \quad \text{in } \mathbb{H} \\ u \in \mathcal{D}^{1,p}\left(\mathbb{H}\right). \end{cases} \tag{D}$$

Naturally, the energy functional corresponding to this half-space problem is simply the restriction of J_∞ to the subspace $\mathcal{D}^{1,p}(\mathbb{H})$.

We remark, perhaps prematurely, that the typical Struwe-type argument involving rescaled translations to extract concentration phenomena will not work in the weighted regime. Indeed, translations are "forbidden" since the equation (A), and its associated energy functional, lose the translation invariance we coveted for $a = b = 0$. In addition, the fact that energy can "bubble away" into unweighted solutions points to the necessity of a transformation, replacing the standard broken rescaling/translation invariance used in the unweighted model case, while accounting for the change in energy space. To this end, before formally stating our decomposition theorem in the case $a = b$, we define a key transform τ^- in terms of a fixed function $\eta \in C_c^\infty(\mathbb{R}^n; [0,1])$ for which $\eta \equiv 1$ in $B(0,1/2)$ and $\eta \equiv 0$ outside $B(0,1)$.

Given $y \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$, we define a transformation $\tau^-_{y,\lambda}$ according to

$$\tau^-_{y,\lambda} h(z) := \begin{cases} \lambda^{-\gamma} h\left(\dfrac{z-y}{\lambda}\right) & \text{if } a = b = 0 \\ \left(\dfrac{\lambda}{|y|}\right)^{-b} \lambda^{-\gamma} h\left(\dfrac{z-y}{\lambda}\right) \eta\left(\dfrac{2(z-y)}{|y|}\right) & \text{if } a = b \neq 0\,. \end{cases}$$

We point to the presence of an additional term, namely $\left(\frac{\lambda}{|y|}\right)^{-b}$, which presents a striking deviation from the classical unweighted model case $a = b = 0$. We shall discuss this transformation in greater detail, following the statement of our next main result, and within Chapter 2. Informally however, one may interpret $\tau^-_{y,\lambda}$ as being a transform that transfers unweighted information into weighted information, while preserving the energy at the limit. We also note that $\tau^-_{y,\lambda}$ reduces to the traditional bubbling transformation used by Struwe [63] and Mercuri-Willem [52] in the unweighted regimes.

We may now state our next main result, which applies to the most difficult case $a = b \neq 0$. Unlike the $a < b$ case, bubbling is *not* exclusive to the origin, and consequently we must impose a suitable smoothness condition on the boundary of Ω, which is consistent with the earlier results applying to the unweighted equation.

Theorem 1.1.2 (Chernysh [14]). *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded C^1-domain containing the origin and let (u_α) be a Palais-Smale sequence for J such that $\lim_{\alpha\to\infty} J(u_\alpha)$ exists. Assume $a = b$ (so that $q = p^*$) and let $\gamma > 0$ be the homogeneity exponent from (1.0.3). Then, there exists a subsequence (u_β) of (u_α) along with*

1. *A solution v_0 of (A).*

2. *Finitely many non-trivial functions $w_1, \dots, w_k$ for the weighted limiting problem (B) and, for each $j = 1, \dots, k$, a sequence $\left(\mu_\beta^{(j)}\right)$ in $(0, \infty)$ such that $\mu_\beta^{(j)} \to 0$ as $\beta \to \infty$.*

3. *Finitely many non-trivial solutions $v_1, \dots, v_l$ for an unweighted limiting problem (C) in $\mathbb{R}^n$ (see below) and, for each $j = 1, \dots, l$, sequences $\left(\lambda_\beta^{(j)}\right)_\beta$ in $(0, \infty)$ and $\left(y_\beta^{(j)}\right)_\beta$ in Ω satisfying, as $\beta \to \infty$,*

$$\lambda_\beta^{(j)} \to 0, \qquad \frac{\left|y_\beta^{(j)}\right|}{\lambda_\beta^{(j)}} \to \infty \qquad \frac{\operatorname{dist}(y_\alpha^{(j)}, \partial\Omega)}{\left|\lambda_\alpha^{(j)}\right|} \to \infty$$

4. *Finitely many non-trivial solutions $v_1^{\mathbb{H}}, \dots, v_{l^{\mathbb{H}}}^{\mathbb{H}}$ for an unweighted limiting problem (D) in the half-space $\mathbb{H}$ (see below) and, for each $j = 1, \dots, l^{\mathbb{H}}$,*

sequences $(\lambda_\beta^{\mathbb{H},(j)})_\beta$ *in* $(0,\infty)$ *and* $(y_\beta^{\mathbb{H},(j)})_\beta$ *in* $\overline{\Omega}$ *satisfying, as* $\beta \to \infty$,

$$\lambda_\beta^{\mathbb{H},(j)} \to 0, \qquad \frac{\left|y_\beta^{\mathbb{H},(j)}\right|}{\lambda_\beta^{\mathbb{H},(j)}} \to \infty, \qquad \frac{\operatorname{dist}(y_\beta^{\mathbb{H},(j)}, \partial\Omega)}{\left|\lambda_\beta^{\mathbb{H},(j)}\right|} \text{ is bounded.}$$

These are such that the following decompositions are true as $\beta \to \infty$

(i) Decomposition in the Energy Space:

$$u_\beta - v_0 - \sum_{j=1}^{k} \left(\mu_\beta^{(j)}\right)^{-\gamma} w_j \left(\frac{z}{\mu_\beta^{(j)}}\right) - \sum_{j=1}^{l} \left[\tau^{-}_{\lambda_\beta^{(j)}, y_\beta^{(j)}} v_j\right] - \sum_{j=1}^{l^{\mathbb{H}}} \left[\tau^{-}_{\lambda_\beta^{\mathbb{H},(j)}, y_\beta^{\mathbb{H},(j)}} V_j^{\mathbb{H}}\right] \to 0$$

in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$. *Here,* $V_j^{\mathbb{H}}$ *are translated rotations of* $v_j^{\mathbb{H}}$.

(ii) Norm Decomposition:

$$\|u_\beta\|^p_{\mathcal{D}^{1,p}(\Omega,0)} \to \|v_0\|_{\mathcal{D}_a^{1,p}(\Omega,0)} + \sum_{j=1}^{k} \|w_j\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} + \sum_{j=1}^{l} \|v_j\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} + \sum_{j=1}^{l^{\mathbb{H}}} \|v_j^{\mathbb{H}}\|_{\mathcal{D}^{1,p}(\mathbb{H})}$$

(iii) Energy Decomposition:

$$J(u_\beta) \to J(v_0) + \sum_{j=1}^{k} J_{0,\infty}(w_j) + \sum_{j=1}^{l} J_\infty(v_j) + \sum_{j=1}^{l^{\mathbb{H}}} J_\infty(v_j^{\mathbb{H}}).$$

Remark 1.1.1. It should be noted that whenever our (PS)-sequence (u_α) is non-negative the functions v_0, w_j, v_j, $v_j^{\mathbb{H}}$ are themselves non-negative. Such an observation is particularly powerful in this case. Indeed, in such a case, thanks to a non-existence result of Mercuri-Willem [52], we may rule out the possibility of solutions to the half-space problem (D). Moreover, the positive solutions to (C) have been classified.

Remark 1.1.2 (The Impact of Symmetry and Boundary Singularities)*.* In the event where Ω is a ball centered at the origin and our (PS)-sequence consists of functions which are radially symmetric about the origin, any loss of compactness must result from bubbles associated to radially symmetric solutions of the limiting weighted problem (B). Indeed, if $a < b$, this is automatic from Theorem 1.1.1. As for the $a = b \neq 0$ regime in the case of radial (PS)-sequences, an adaptation of the proof of Theorem 1.1.2 (see for example the argument in Proposition 3.4.2 of Chapter 3) shows that any loss of compactness must take place at the origin, in the form of a weighted bubble (i.e. bubbles arising from solutions to (B)) and as such recover the bubble tree decomposition from Theorem 1.1.1. Thus, in the case of a radial (PS)-sequence, and for all valid parameters, we may dispense with the limiting problems (C)-(D) in the statement of Theorem 1.1.2. More generally, if B denotes an open ball centered at the origin, this bubble tree reduction remains true for any (PS)-sequence (u_α) in $\mathcal{D}^{1,p}_a(B, 0)$ such that for almost all $x \in B$:

$$|u_\alpha(gx)| = |u_\alpha(x)|$$

for all $\alpha \in \mathbb{N}$ and $g \in G$, where $G \leq O(n)$ is a subgroup such that $|\mathrm{Orb}_G(x)|$ is infinite for all $x \neq 0$.

Finally, although our main results are stated for bounded domains Ω containing the origin, we expect that Theorems 1.1.1-1.1.2 extend (after an adaptation of their proofs) to the case $0 \in \partial\Omega$, provided we replace the weighted problem (B) with its analogue posed in a half-space. Note that, even in the case $a < b$, we would require that $\partial\Omega$ be at least C^1 near the point 0. We remark that our assertion regarding the decomposition in this case, and the regularity assumption on $\partial\Omega$ near 0, is entirely consistent with the results obtained by Ghoussoub-Kang [40] for the Hardy-Sobolev case $p = 2$ and $a = 0$.

Restricting ourselves to the special case where $p = 2$ and $a = b = 0$, Theorem 1.1.2 reduces to precisely the result first obtained by Struwe [63]. Indeed, in the unweighted case with the Laplacian, a Pohozaev identity shows that the half-space problem (D) admits no non-trivial solutions, and consequently we may only find bubbles solving the problem (1.1.2). For general $1 < p < n$ with $a = b = 0$, Theorem 1.1.2 reduces to the result established by Mercuri-Willem [52] for Palais-Smale sequence whose negative part converges strongly to 0 in $L^{p^*}(\Omega)$.

However, we should observe that the weighted regime $a = b \neq 0$ carries surprising caveats, some of which are strikingly different from the unweighted model cases. As we emphasized earlier, Theorem 1.1.2, in contrast to Theorem 1.1.1 and the unweighted cases studied by Struwe [63] and Mercuri-Willem [52], may produce bubbles corresponding to three problems, specifically (B), (C), (D). Nevertheless, it is Theorem 1.1.2 that generalizes the model case $a = b = 0$.

Perhaps an even more notable facet of Theorem 1.1.2 is not simply the fact that we have three possible limiting problems, but rather the surprising property that bubbles may arise in both weighted and in unweighted energy spaces. In truth, it is the presence of bubbles that "feel" the weight, while others remain unaffected, along with the broken translation invariance of the problem, that contributes the most significant difficulty of the proof when $a = b \neq 0$ in the parameters. What is more, it is precisely due to this interplay between weighted and unweighted energies that we require a transform $\tau^{-}_{y,\lambda}$, which enables the 'transfer of information" between the unweighted energy space $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and its weighted counterpart $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$. In said transform, we observe the presence of a concentration factor which vanishes in the unweighted model case $a = b = 0$: a power of $\lambda/|y|$ accounting for the effect of the weighted measures $|x|^{-ap}\,\mathrm{d}x$ and $|x|^{-bq}\,\mathrm{d}x$, which are controlled via a cutoff function. Of course, it is important to observe that the $\lambda/|y|$ ratio vanishes when $a = b = 0$ and the cutoff is unnecessary in this unweighted case, at which point we recover the classical transform used by Struwe [63] and Mercuri-Willem [52].

Parameter:	**Core Problem Case**	**Possible Limit Problems**
$a = b = 0$	Critical p-Laplace problem	(C) , (D)
$a < b$	CKN problem (A) with $a < b$	(B)
$a = b \neq 0$	CKN problem (A) with $a = b \neq 0$	(B), (C) , (D)

Table 1.1: Compactness profile by weight parameters.

Classification of Solutions. Our understanding of the loss of compactness phenomenon presented in Theorems 1.1.1, 1.1.2 may be improved by characterizing solutions to problems (B), (C) and (D). As such, we provide some historical context and recent results, particularly with regards to classifications of solutions to problem (B) and its unweighted analogue (C). In fact, early classifications were for extremal functions of (CKN) with $\Omega = \mathbb{R}^n$. Without the weights (i.e. for $a = b = 0$), this reduces to the Gagliardo-Nirenberg-Sobolev inequality, and the best constant is known by works of Aubin [3] and Talenti [66]; though computations of this constant date as far back as Rodemich's seminar [57]. In this unweighted case, Talenti [66] exhibited a family of (radially symmetric) functions achieving the Caffarelli-Kohn-Nirenberg inequality in $\mathbb{R}^n$. For $p = 2$ and $n \geq 3$, the existence of radially symmetric functions minimizing (CKN) in $\mathbb{R}^n$ was determined by Lieb [47] when $a = 0$. In addition, an explicit form for said best constant was established therein. Furthermore, for $p = 2$, Chou and Chu [16] determined the best constant in the Caffarelli-Kohn-Nirenberg inequality when $a \geq 0$ and categorized the minimizing functions when $a > 0$. Considering instead $a < 0$, Horiuchi [45] showed that, when $p = 2, n \geq 3$ and $a = b < 0$, the best constant in the Caffarelli-Kohn-Nirenberg inequality is precisely the Sobolev constant, and equality cannot be achieved by any function in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$. In this same paper, Horiuchi also determined the best constant of the Caffarelli-Kohn-Nirenberg inequality for $1 < p < n/(1 + a - b)$, provided $a \geq 0$ and $b < n/q$.

In recent years, positive solutions to the unweighted entire problem (C) have been fully classified and have the form

$$x \mapsto \left[\frac{\left(\mu n \left(\frac{n-p}{p-1} \right)^{p-1} \right)^{1/p}}{\mu + |x - x_0|^{\frac{p}{p-1}}} \right]^{\frac{p}{q-p}}, \qquad \mu > 0, x_0 \in \mathbb{R}^n. \tag{1.1.5}$$

This classification was obtained in the case $p = 2$ by Obata [53] and Caffarelli, Gidas and Spruck [8]; they showed, via the method of moving planes, that all positive solutions to (C) are radial and thus determined an explicit form for said solutions. Furthermore, Damascelli, Merchán, Montoro and Sciunzi [25] utilized a previous classification due to Guedda and Veron [41] to extend this result to the range $\frac{2n}{n+2} \leq p < 2$. Similarly, Vétois [68] improved the range to $1 < p < 2$ by exploiting a decay result of Damascelli-Ramaswamy [26]. Finally, the case $2 < p < n$ was handled by Sciunzi [59] using an a-priori estimate of Vétois [68]. We mention that, for ranges of p depending on n, this classification is also known for solutions that are *not* assumed to have finite energy (i.e. u need not belong to $\mathcal{D}^{1,p}(\mathbb{R}^n)$ a priori), see Catino, Monticelli and Roncoroni [10], Ou [54] and Vétois [69] for more detail. Furthermore, we note that Farina-Mercuri-Willem [34] extended these classifications to (possibly sign-changing, a priori) radial solutions by showing that any non-trivial radial solution to (C) is necessarily of the form (1.1.5), up to multiplication by ± 1.

Similar classification results are known for the weighted analog of (C), i.e. problem (B). Indeed, in some cases, positive solutions to problem (B) are known to be of the form

$$x \mapsto \left[\frac{\left(\frac{\mu n (n-p(1+a))^p}{(n-p(1+a-b))(p-1)^{p-1}} \right)^{1/p}}{\mu + c\,|x|^{\frac{(n-p(1+a))}{(n-p(1+a-b))}\frac{p(1+a-b)}{p-1}}} \right]^{\frac{p}{q-p}}, \qquad \mu > 0 \tag{1.1.6}$$

We note that, if $a = b = 0$, the weighted measures $|x|^{-ap}\,\mathrm{d}x$ and $|x|^{-bq}\,\mathrm{d}x$ are translation invariant; in particular the functions in (1.1.6) cover all possible solutions of (B), up to translation. Put otherwise, in the case $a = b = 0$ there is no distinction between problems (B) and (C) so all positive solutions are given by equation (1.1.5); we omit this case from the reminder of our discussion. When $a = 0$, generalizing a result of Guedda-Veron [41] to allow for $b \neq 0$, Ghoussoub and Yuan [39] showed that every positive radial solutions of (B) is of the form (1.1.6). Considering only $p = 2$, Dolbeault and Esteban [31] show that all positive (not a-priori assumed radial) solutions in the non-limiting case $a < b$ are as in (B), provided either $a \geq 0$ or

$$a < 0 \quad \text{and} \quad b \leq \gamma \left(\frac{n}{p\sqrt{\gamma^2 + n - 1}} - 1 \right).$$

Moreover, for dimensions $n \geq 3$, Ciraolo and Corso [17], utilizing a-priori asymptotic estimates of Shakerian-Vétois [60], demonstrated that this classification extends to the whole range $1 < p < n$ in the limit case $a = b \geq 0$ as well as a wider range of possible values for a, b provided $p < n/2$. Building upon the result of Ciraolo-Corso [17] and again utilizing estimates from Shakerian-Vétois [60], Le and Le [46] showed that, when $a = 0$, all positive solutions of (B) are radial and therefore classified by Ghoussoub-Yuan [39].

We remark that all classification results (for positive solutions) discussed thus far yield radial solutions. However, certain minimizers of the Caffarelli-Kohn-Nirenberg inequality are known to break symmetry for some values of a, b, p; see [11, 7, 35, 62, 32, 15] for details.

Finally, we also point out that the stability of the Caffarelli-Kohn-Nirenberg inequality, which has close relation to quantitative versions of Struwe decompositions, has received significant attention recently. Indeed, questions of this type have been studied in the papers of Ciraolo-Figalli-Maggi [18], Figalli-Glaudo [38], Deng-Sun-Wei [29], and Wei-Wu [70].

1.2 Nodal Solutions with Prescribed Symmetry

The remainder of this thesis delves into questions of existence, multiplicity, and symmetry prescription for nodal solutions of the entire Caffarelli-Kohn-Nirenberg problem

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bq}|u|^{q-2}u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0) \end{cases} \tag{B}$$

Questions of this type for quasilinear critical elliptic equations have a long standing and rich history. In fact, recently in 2025, Clapp-Vicente-Benítez [21] constructed entire nodal solutions for the critical p-Laplace equation (C) for an infinite number of distinctive symmetry types. As far as the author of this thesis can tell, this paper of Clapp-Vicente-Benítez [21] contains the first such result for the pure critical problem (C) (alongside an investigation of associated competitive systems). We also remark that, through a similar methodology, Clapp-Rios [24] found (in 2017) the first nodal solutions for (C), by considering a finite number of distinct symmetry types. In both these aforementioned works, the symmetries considered act exclusively on four real coordinates.

It should also be observed that, when taking $p = 2$ and $a = b = 0$, the weighted problem (B) reduces to the Yamabe problem (1.1.2). Not only does this problem have important geometric significance, it arises as the natural limit profile bubbles satisfy when $p = 2$ and $a = b = 0$ (Struwe [63]). In this particular setting, there has also been a great deal of continued interest in the existence of nodal solutions, particularly when it comes to exhibiting those satisfying new symmetries. In this

Yamabe context, a key reference is the seminal work of Ding [71], wherein it was proven that the critical problem (1.1.2) admits infinitely many distinct solutions which change sign. In fact, to be more precise, Ding constructed a sequence of functions (u_α) in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, each solving (1.1.2), for which

$$\|u_\alpha\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \, \mathrm{d}x \to \infty,$$

as $\alpha \to \infty$. In his proof, Ding exploited the invariance of the Yamabe problem under Möbius transformations, an approach that is not appropriate for even the unweighted p-Laplacian since this operator is *not* conformally invariant. Further significant attention has also been given to finding new solutions to the Yamabe problem which change sign, i.e. nodal solutions to (1.1.2) which do not arise from Ding's procedure in [71]. For important references on such questions, we mention the papers of Pino-Musso-Pacard-Pistoia [27]-[28]. Further highlighting the difficulties presented by even the unweighted p-Laplacian, we observe that the authors of [27] utilize the Lyapunov-Schmidt reduction method within their paper, an approach which does not readily generalize to the weighted p-Laplacian, being that the resulting linearized operator is poorly understood for even the unweighted p-Laplace operator. With regards to the case of the sphere, we point to works of Fernández-Palmas-Petean [36], Fernández-Petean [37], and Medina-Musso [51]. It should be noted that Ding's result was not anticipated by all; in fact, there had been some who had previously attempted to prove that *every* solution of (1.0.5) with finite energy (i.e. belonging to $\mathcal{D}^{1,2}(\mathbb{R}^n)$) was radial and hence classified by the Talenti bubbles. Of course, we now know this to be an impossibility. Recent work in finding new solutions to the Yamabe problem has also been done by Clapp-Faya-Saldaña [23]. As for weighted equations of the Caffarelli-Kohn-Nirenberg type with subcritical terms, the literature is much more sparse. However, we observe that Szulkin-Waliullah [64] constructed entire sign-changing solutions when the parameters in (1.0.1) satisfy $0 \leq a < b$. Similar approaches have also been used in the fractional case (see, for instance, Hernández-Santamaría-Saldaña [43] and Zhang-Ma-Zheng [75]).

In much of the literature addressing questions of existence for nodal solutions to these equations, the potential symmetries such functions can satisfy are a central and recurring theme, both in terms of qualitative analysis and proof techniques. For example, the solutions constructed by Ding [71] are invariant under the action of an $O(k) \times O(\ell)$ subgroup of $O(n)$, while the solutions constructed by Clapp-Rios [24] and Clapp-Vicente-Benítez [21] satisfy carefully selected rotation, reflection, and cycling (anti)-symmetries.

Our aim, both in this section and in Chapter 3, is to establish the existence of sign-changing solutions to the *entire* critical Caffarelli-Kohn-Nirenberg problem

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}\left|\nabla u\right|^{p-2}\nabla u\right) = |u|^{q-2}\,u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0) \end{cases} \tag{1.2.1}$$

across the full range of parameters described by (1.0.1). In addition, we seek to put forth a broad, formal category, of symmetries for which there exists a sign-changing solution of (1.2.1) belonging to the aforementioned symmetry class. This symmetry category will extend those introduced by Clapp-Rios [24] and Clapp-Vicente-Benítez [21]. We point out that our category of symmetries is selected for two specific reasons: first, it provides a framework under which one may distinguish the solutions by their symmetry types, while being broad enough to enable the prescription of an infinite variety of symmetry types to sign-changing solutions.

In order to present our main result, we must first define some conditions. Henceforth, given a dimension $n \geq 4$, we shall write $k := \left\lfloor \frac{n}{4} \right\rfloor - 1$, fix an integer $\alpha \geq 0$, and denote by $\mathfrak{m} = (m_1, \ldots, m_k)$ a tuple of non-negative integers. We define the conditions:

$$0 < 2\chi_\alpha + \sum_{j=1}^{k} m_j(j+1) \leq n/2, \tag{1.2.2}$$

$$2\chi_\alpha + \sum_{j=1}^{k} m_j(j+1) \neq (n-1)/2. \tag{1.2.3}$$

Here, $\chi_\alpha = 0$ if $\alpha = 0$ and $\chi_\alpha = 1$ otherwise. If $\alpha = 0$, we require that $\mathfrak{m} \neq \mathbf{0}$. It should be noted that (1.2.3) is important to the case $a = b \neq 0$ and is the reason for the condition $n \neq 5$ in Theorem 1.2.1 stated below. The pair $(\alpha, \mathfrak{m})$ will serve as a *barcode* identifying a precise symmetry configuration. However, speaking informally, one can interpret an $(\alpha, \mathfrak{m})$-symmetric function as an entire function $\mathbb{R}^n \to \mathbb{R}$, vanishing at the origin, for which $\alpha \geq 0$ describes the number of "pins" or "pinwheels" the function possesses in $\mathbb{R}^4$, while $\mathfrak{m}$ dictates a more complex symmetry structure, involving invariance under certain synchronous rotations, and sign-changes with respect to special reflections and coordinate cycling. We refer the reader to Chapter 3 for a precise definition of an $(\alpha, \mathfrak{m})$-symmetric function. However, to help ground the notion of an $(\alpha, \mathfrak{m})$-symmetry in the current body of literature, we point out that every solution obtained by Clapp-Rios [24] has an $(0, (j, \mathbf{0}))$-symmetry. Furthermore, although the construction implemented by Clapp-Vicente-Benítez [21] provides no information on the symmetries of the solutions beyond 4-real coordinates, we are certain to produce solutions different from theirs.

Fix a dimension $n \geq 4$. Given two tuples $\mathfrak{m} = (m_1, \ldots, m_k)$ and $\mathfrak{n} = (n_1, \ldots, n_k)$, we say that $\mathfrak{m} \lesssim \mathfrak{n}$ provided there exists $\ell \in \{1, \ldots, k\}$ such that

$$m_j = n_j \text{ if } j < \ell, \quad m_\ell \leq n_\ell, , \quad \text{and} \quad m_\ell = 0 \text{ if } j > \ell.$$

It is not hard to check that $\mathfrak{m} = \mathfrak{n}$ if and only if $\mathfrak{m} \lesssim \mathfrak{n}$ and $\mathfrak{n} \lesssim \mathfrak{m}$. In fact, if $\mathfrak{m} \not\lesssim \mathfrak{n}$ and $\mathfrak{n} \not\lesssim \mathfrak{m}$, then there exist distinct indices $i, j \in \{1, \ldots, k\}$ such that $m_i, n_j \neq 0$. Therefore, we are free to define

$$\gcd(\mathfrak{m}, \mathfrak{n}) := \begin{cases} 1 & \text{if } \mathfrak{m} \lesssim \mathfrak{n} \text{ or } \mathfrak{n} \lesssim \mathfrak{m} \\ \max\{\gcd(i+1, j+1) : m_i, n_j \neq 0\} & \text{otherwise.} \end{cases} \tag{1.2.4}$$

The main theorem of our second chapter is the following existence result for nodal solutions of (1.2.1).

Theorem 1.2.1 (Chernysh [13], 2025)**.** *Fix a dimension $n \geq 4$, let $\alpha \geq 0$ an integer, and $\mathfrak{m}$ be a tuple of non-negative integers such that $(\alpha, \mathfrak{m})$ satisfies (1.2.2).*

1. *When $a < b$ or $a = b = 0$, there exists a non-trivial sign-changing solution $u_{\alpha,\mathfrak{m}}$ to (1.2.1) which is $(\alpha, \mathfrak{m})$-symmetric.*

2. *When $a = b \neq 0$, $n \neq 5$, and $(\alpha, \mathfrak{m})$ satisfies condition (1.2.3), there exists a non-trivial sign-changing solution $u_{\alpha,\mathfrak{m}}$ to (1.2.1) which is $(\alpha, \mathfrak{m})$-symmetric.*

Moreover, for any $(\beta, \mathfrak{n})$ *satisfying* (1.2.2)*:*

(i) if $\alpha \neq \beta$ *then* $u_{\alpha,\mathfrak{m}} \neq u_{\beta,\mathfrak{n}}$*,*

(ii) if $\alpha = \beta$ *and* $\mathfrak{m} \neq \mathfrak{n}$*, then* $u_{\alpha,\mathfrak{m}} \neq u_{\beta,\mathfrak{n}}$ *provided* $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

A few brief remarks are in order regarding the contents of Theorem 1.2.1. First, to the best of the author's knowledge, Theorem 1.2.1 is the first result providing the existence of entire nodal solutions to (1.2.1) for the full parameter range (1.0.1). In fact, Theorem 1.2.1 seems to be the first result establishing the existence of even a single sign-changing solution to (1.2.1) when $p \neq 2$ and $a < 0$. What is more, this result is established using a single approach, which applies to the full range of parameters (1.0.1).

Aside from the importance of answering the question of existence, the main strength of Theorem 1.2.1 is its ability to generate a plethora of nodal solutions to (1.2.1), while providing a wide class of symmetries one can prescribe to nodal solutions, in such a way that the resulting functions are readily and systematically distinguishable by their symmetry-types. In fact, by fixing $\mathfrak{m}$ and allowing $\alpha \geq 0$ to vary, we see that there exist sign-changing solutions to (1.2.1) for an *infinite number* of distinct symmetry configurations. Additionally, for a fixed integer $\alpha \geq 0$, there is a multitude of admissible tuples $\mathfrak{m}$ for which $(\alpha, \mathfrak{m})$ satisfies (1.2.2) for which the resulting symmetry classes are distinct. Certainly, if we restrict ourselves to tuples $\mathfrak{m} = (m_1, \dots, m_k)$ for which $m_j \neq 0$ only when $(j+1)$ is prime, then the resulting $(\alpha, \mathfrak{m})$-symmetry classes are automatically mutually exclusive by the second part of Theorem 1.2.1. In such cases, we shall say that $\mathfrak{m}$ is *prime*, i.e. that $m_j = 0$ only when $(j+1)$ is prime. Famously, a beautiful result of Ramanujan [56] then asserts that, for a fixed $\alpha \geq 0$, the possible admissible choices prime $\mathfrak{m}$ such that $(\alpha, \mathfrak{m})$ satisfies (1.2.2)-(1.2.3) is asymptotically in the order of

$$\exp\left(\frac{\pi\sqrt{2n}}{\sqrt{3\ln(n/2)}}\right). \tag{1.2.5}$$

Returning briefly to the topic of symmetry prescription, we point out that, as far as the author is aware, Theorem 1.2.1 provides new solutions to the critical p-Laplace problem (C) and, also, to the Yamabe problem (1.1.2).

Before we discuss any of the proof elements, we ought to acknowledge the interesting distinctions between the cases $a < b$, $a = b = 0$, and $a = b \neq 0$. As one can observe from the first part of the theorem, in the cases $a < b$ and $a = b = 0$, we can guarantee the existence of an $(\alpha, \mathfrak{m})$-symmetric nodal solution for any $(\alpha, \mathfrak{m})$ satisfying condition (1.2.2). On the other hand, for $a = b \neq 0$, we require that $(\alpha, \mathfrak{m})$ satisfies the auxiliary condition (1.2.3), which is precisely what gives rise to the exclusion of dimension $n = 5$ in this case. The condition (1.2.3) is effectively a group symmetry constraint which serves to handle the complex concentration phenomena exhibited by the Struwe-type decomposition in the case $a = b \neq 0$ (see Theorem 1.1.2). Informally, its purpose is to prohibit the formation of bubbles solving the unweighted limiting problem, ensuring that our variational methods produce a solution to the weighted problem of interest. In the unweighted regime $a = b = 0$, a weaker condition (see Clapp-Rios [24]) is sufficient because of the translation invariance of the problem, which is broken by the presence of weights. As for the case $a < b$, we need no such condition at all by virtue of the Struwe-type decomposition forcing all loss of compactness to take place at the origin (see Theorem 1.1.1), thereby automatically locking potential bubbling into the category of weighted solutions. We will elaborate more on these technical points in Chapter 3, where we produce the proof of Theorem 1.2.1.

1.3 Thesis Organization

Following this chapter, we devote the entirety of Chapter 2 to the proof of our Struwe-type decompositions. Here, we will recall important properties regarding the weighted homogeneous Sobolev space $\mathcal{D}_a^{1,p}(\Omega, 0)$, and set up the problem (A) from a variational standpoint. This Chapter is largely technical, as it requires a detailed blow-up analysis and a new rescaling law, to handle the complex balanced setting $a = b \neq 0$, where the presence of a weight at the origin competes with the "criticality" introduced by the critical exponent $q = p^*$. Of particular note here is the fact that loss of compactness looks very different in each of the three cases $a < b$, $a = b = 0$, and $a = b \neq 0$.

Briefly addressing the proofs of Theorems 1.1.1-1.1.2, it is instructive to observe that the transformation rules employed within our strategy deviate significantly from those used by Struwe [63] and Mercuri-Willem [52]. Speaking informally, in the case $a < b$, we require a transform that accounts for the lack of translation invariance introduced by the aforementioned weighted measures. We also note that the proof of Theorem 1.1.1 coincides with a sub-argument in the $a = b$ case and, on that account, constitutes a key facet of the global picture. In addition to this, since when $a = b$ we obtain bubbles derived from solutions belonging to distinct energy spaces, we shall also requires a transform $\tau_{y,\lambda}$ from the weighted space $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ to $\mathcal{D}^{1,p}(\mathbb{R}^n)$, which acts as a "pseudo-inverse" to $\tau_{y,\lambda}^-$ appearing in Theorem 1.1.2; both $\tau_{y,\lambda}$ and $\tau_{y,\lambda}^-$ will be discussed in greater depth within §2. We note that $\tau_{y,\lambda}$ will also be central in the proof of Theorem 1.1.1, where it is used to rule out bubbling away from the origin.

Chapter 3 turns instead towards the construction of nodal (i.e. sign-changing) solutions to the problem (A), for $\Omega = \mathbb{R}^n$. Here, we combine the conclusions of the Struwe-type decomposition results, obtained in Chapter 2, together with carefully chosen subgroups of $O(n)$, to build non-trivial sign-changing solutions across the full range of parameters (1.0.1). Furthermore, we exhibit a symmetry category where one can systematically prescribe certain symmetries, and distinguish the resulting functions according to a "barcode"-scheme. Of particular note here is that, to the best of the author's knowledge, the main result of Chapter 3 is the first theorem guaranteeing the existence of entire sign-changing solutions to (A) (for $\Omega = \mathbb{R}^n$) for the full range of parameters.

Chapter 2

A Struwe-type Decomposition

The purpose of this chapter is to develop the generalized Struwe-type decompositions described in Chapter 1, for the Caffarelli-Kohn-Nirenberg problem (A). As mentioned previously, such an endeavour requires a basic theory of the homogeneous weighted Sobolev space $\mathcal{D}_a^{1,p}(\Omega, 0)$. We shall first discuss this Sobolev space, and related weighted Lebesgue spaces. Being that the purpose of this chapter is to prove Theorems 1.1.1-1.1.2, we merely recall, without providing any proofs, the statements of important topological properties (e.g. Rellich-Kondrachov and Riesz representation analogues) about the energy space $\mathcal{D}_a^{1,p}(\Omega, 0)$ that will be invoked liberally during the remainder of this chapter. We point the reader towards Chapter 2 of the author's masters thesis Chernysh [12] for proofs, and a more elaborate discussion of $\mathcal{D}_a^{1,p}(\Omega, 0)$.

Following this, we introduce technical tools used to obtain the asymptotic energy expansions present within the statements of Theorems 1.1.1-1.1.2. This includes a truncation operator introduced to obtain the almost everywhere convergence of the gradients, as well as the new rescaling laws used to handle concentration phenomena in the weighted regime. In this $a < b$ regime, this rescaling law takes the form of a pure rescaling centered at a "displaced singularity". However, in the critical regime $a = b \neq 0$, even this rescaling law is insufficient, as the "displaced singularity" may slide off to infinity. In such a case, we require an additional conditional rescaling law (this transform $\tau_{y,\lambda}^-$) appearing in the statement of Theorem 1.1.2), which serves to pass information between the unweighted space

$\mathcal{D}^{1,p}(\mathbb{R}^n)$ and the weighted Sobolev space $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$. As these adapted rescaling laws are fundamental to the analysis we conduct, we devote the first few sections of this chapter to studying the properties of Palais-Smale sequences (for the energy functional J) and produce estimates for our rescaling laws. Following this, we briefly analyze the convergence of rescaled domains in $\mathbb{R}^n$, in order to rigorously handle the possibility of bubbles solving the half-space problem (D). We remark that this theory is nothing special or new; in fact, considering the possibility of an *unweighted* half-space problem is a necessary component of the proof even in the model case $a = b = 0$ (i.e. such an argument appears in both Struwe [63] and Mercuri-Willem [52]). For the sake of completeness and the quality of exposition, we nonetheless opt to precisely develop the required notions to handle this case.

Finally, we establish key lemmas providing the necessary convergence conditions required for the energy decomposition, after which we address the proofs of Theorems 1.1.1-1.1.2, beginning with the former. Although the $a < b$ case (Theorem 1.1.1) was first established during the author's masters thesis (see Chernysh [12]), omitting the proof would not serve to shorten the length of this manuscript. Indeed, the proof of the $a < b$ case arises as a natural sub-argument within the proof of the critical regime $a = b$ (Theorem 1.1.2). In fact, because the proof when $a = b \neq 0$ is even more technical than the $a < b$ regime, it is the author's belief that separating the argument into two portions (as we do) significantly improves the quality of exposition. Moreover, presenting it in this manner provides a more accurate description of the global problem, as it highlights the key differences between the three cases $a < b$, $a = b = 0$, and $a = b \neq 0$.

For the remainder of this chapter, unless stated otherwise, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain containing the origin. Furthermore, we always assume $n \geq 2$ and that the parameters p, a, b satisfy (1.0.1).

2.1 Homogeneous CKN-Sobolev Spaces

The focus of this section is to provide a concrete framework for the weighted homogeneous Sobolev space $\mathcal{D}_a^{1,p}(\Omega, 0)$. Although we have already defined this space in the introduction, it will be convenient to consider the same space with a "shifted singularity". In addition, our definition involved a completion process, which one might naturally wish to replace with an explicit identification.

To this end, let $U \subseteq \mathbb{R}^n$ be an open set and fix a point $x_0 \in U$. We define $\mathcal{D}^{1,p}_a(U, x_0)$ to be the completion of $C^\infty_c(U)$ under the norm

$$\|u\|_{\mathcal{D}^{1,p}_a(U,x_0)} := \left(\int_U |\nabla u(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \right)^{1/p}.$$

The base-point x_0 will be called the *singularity point*. We also introduce the following notation for weighted Lebesgue spaces, which are intrinsic to the space $\mathcal{D}^{1,p]}_a(U, x_0)$:

$$\|u\|^q_{L^q_b(U,x_0)} := \int_U |u|^q \, |x - x_0|^{-bq} \, \mathrm{d}x, \quad \|v\|^p_{L^p_a(U,x_0)} := \int_U |v|^p \, |x - x_0|^{-ap} \, \mathrm{d}x.$$

These expressions induce norms on the weighted Lebesgue spaces, denoted by $L^q_b(U, x_0)$ and $L^p_a(U, x_0)$, consisting (respectively) of (Lebsegue measurable) functions $u, v : U \to \mathbb{R}$ such that

$$\int_U |u(x)|^q \, |x - x_0|^{-bq} \, \mathrm{d}x < \infty, \quad \text{and} \quad \int_U |v(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x < \infty.$$

In this notation, the Caffarelli-Kohn-Nirenberg inequality (CKN) is thus given by

$$C_{a,b} \, \|w\|^p_{L^q_b(\mathbb{R}^n,x_0)} \leq \|\nabla w\|^p_{L^p_a(\mathbb{R}^n,x_0)}, \quad \forall w \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0) \tag{2.1.1}$$

where $C_{a,b} = C_{a,b}(n, p)$ denotes the sharp constant that will henceforth be called a CKN-constant.

Remark 2.1.1. As mentioned in Chapter 1, when $a = b = 0$, we shall omit the presence of weight terms entirely and only write $\mathcal{D}^{1,p}(U)$. In this case, with $U = \mathbb{R}^n$, available literature includes, for instance, Willem [74, §7.2], wherein it is shown that this homogeneous Sobolev space is precisely

$$\mathcal{D}^{1,p}(\mathbb{R}^n) = \left\{ u \in L^{p^*}(\mathbb{R}^n) : \nabla u \in L^p(\mathbb{R}^n) \right\}.$$

For the purposes of this monograph, we shall require results from Sobolev theory in the context of the weighted space $\mathcal{D}^{1,p}_a(U, x_0)$, which we recall below.

The reader may consult Chernysh [12, Chapter 2] for more details on these results. Descriptively, every function $u \in \mathcal{D}^{1,p}_a(U, x_0)$ belongs to $L^q_b(U, x_0)$, whereas ∇u exists in the weak sense on $U \setminus \{x_0\}$ such that $\nabla u \in L^p_a(U \setminus \{x_0\}, x_0)$ (see Chernysh [12, Theorem 2.3.1]). Moreover, when $a \geq 0$, we can assert that ∇u exists in the weak sense on the *whole* of U ([12, Corollary 2.3.3]). We further note that weak gradients vanish on level sets. That is, for every $c \in \mathbb{R}$,

$$\nabla u \equiv 0 \text{ almost everywhere on } \{x \in \mathbb{R}^n : u(x) = c\}\,.$$

We also recall that, under condition (1.0.1), $\mathcal{D}^{1,p}_a(U, x_0)$ is always a reflexive Banach space. In fact, every bounded sequence in $\mathcal{D}^{1,p}_a(U, x_0)$ admits a subsequence converging weakly and almost everywhere on the set U (see [12, Theorem 2.5.1]).

Fortunately, several key analytic results for traditional Sobolev spaces have known analogues in $\mathcal{D}^{1,p}_a(U, x_0)$, e.g. the Rellich-Kondrachov embedding theorem. In particular, the natural embeddings

$$\begin{aligned}
&\mathcal{D}^{1,p}_a(U, x_0) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^n, x_0)\\
&\mathcal{D}^{1,p}_a(U, x_0) \hookrightarrow L^r(U, |x - x_0|^{-\theta}), \quad 1 \leq r < q,\, \theta \in (-\infty, bq]
\end{aligned}$$

are compact (see [12]).

Finally, we require a complete description of the continuous linear functionals on $\mathcal{D}^{1,p}_a(U, x_0)$ for the proofs of our Struwe-type decompositions (specifically, to show the bubbles are non-trivial). Henceforth, we shall write $\mathcal{D}^{-1,p'}_a(U, x_0)$, with $p' := p/(p-1)$ being the Hölder conjugate exponent of p, to denote the topological dual of $\mathcal{D}^{1,p}_a(U, x_0)$. An extension of the Riesz-type representation theorem (see Chernysh [12, Theorem 2.5.3]) asserts that continuous linear functionals $\varphi \in \mathcal{D}^{-1,p'}_a(U, x_0)$ are of the form

$$\varphi(u) = \int_U (\nabla u, \mathbf{g})\, |x - x_0|^{-ap}\, \mathrm{d}x, \qquad \forall u \in \mathcal{D}^{1,p}_a(U, x_0) \tag{2.1.2}$$

for some $\mathbb{R}^n$-valued function $\mathbf{g} \in L^{p'}_a(\mathbb{R}^n, x_0)$. We reiterate once more that, $(\cdot, *)$ denotes the usual inner-product on $\mathbb{R}^n$. As is convention, if $a = b = 0$, the topological dual of $\mathcal{D}^{1,p}(U)$ is simply denoted $\mathcal{D}^{-1,p'}(U)$.

2.1.1 Pointwise Convergence of the Gradients

We have noted that bounded sequences in $\mathcal{D}_a^{1,p}(U, x_0)$ have subsequences that converge weakly and almost everywhere on U. In this subsection, we give conditions under which we can find a subsequence whose *gradients* converge pointwise. In short, we establish a weighted generalization of Theorem 3.3 in Mercuri-Willem [52].

Let us first define the Lipschitz continuous function $T : \mathbb{R} \to \mathbb{R}$ by the rule

$$T(x) := \begin{cases} x & \text{if } |x| \le 1, \\ \frac{x}{|x|} & \text{if } |x| > 1. \end{cases} \tag{2.1.3}$$

By considering suitable mollifications of T, it is straightforward to see that, given $u \in \mathcal{D}_a^{1,p}(U, x_0)$, $T(u) \in \mathcal{D}_a^{1,p}(U, x_0)$.

Proposition 2.1.1. *Let (u_α) be a bounded sequence in $\mathcal{D}_a^{1,p}(U, x_0)$ converging pointwise almost everywhere to $u \in \mathcal{D}_a^{1,p}(U, x_0)$ as $\alpha \to \infty$. Let T be given by (2.1.3) and assume that (U_k) is an increasing sequence of open subsets of U such that $\bigcup_{k \ge 1} U_k = U$. Assume further that*

$$\lim_{\alpha \to \infty} \int_{U_k} \left(\left[|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u \right], \nabla \left[T(u_\alpha - u) \right] \right) \mathrm{d}\mu = 0 \tag{2.1.4}$$

for each $k \ge 1$, where $\mathrm{d}\mu = |x - x_0|^{-ap} \, \mathrm{d}x$. Then, there exists a subsequence (u_β) such that $\nabla u_\beta \to \nabla u$ almost everywhere on U.

Proof. We adapt an argument from Szulkin-Willem [65], to show that, given $k \in \mathbb{N}$, up to a subsequence, $\nabla u_\alpha \to \nabla u$ almost everywhere on U_k.

Fix $k \in \mathbb{N}$ and, for each $\alpha \in \mathbb{N}$, consider the set

$$E_\alpha^{(k)} := \{ x \in U_k : |u_\alpha(x) - u(x)| \le 1 \}.$$

Then, (2.1.4) can be written as

$$\lim_{\alpha \to \infty} \int_{E_\alpha^{(k)}} \left(\left[|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u \right], \left[\nabla u_\alpha - \nabla u \right] \right) |x - x_0|^{-ap} \, \mathrm{d}x = 0. \tag{2.1.5}$$

Since the integrand is non-negative and $\mathbf{1}_{E_\alpha^{(k)}}(x) \to 1$ for almost every $x \in U_k$, passing to a subsequence if necessary, we see that

$$\left(\left[|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u\right], [\nabla u_\alpha - \nabla u]\right) \to 0$$

pointwise almost everywhere on U_k. Appealing classical results (see, for instance, Szulkin-Willem [65, Lemma 2.1]), it follows that $\nabla u_\alpha \to \nabla u$ pointwise a.e. on U_k. Finally, a diagonal argument completes the proof. □

Corollary 2.1.2. *Let (u_α) be a bounded sequence in $\mathcal{D}_a^{1,p}(U, x_0)$ converging pointwise almost everywhere to $u \in \mathcal{D}_a^{1,p}(U, x_0)$ as $\alpha \to \infty$. Under the assumptions of Proposition 2.1.1, there exists a subsequence (u_β) with the following properties:*

(1) $\nabla u_\beta \to \nabla u$ *pointwise almost everywhere on* U*;*

(2) $\lim_{\beta \to \infty} \left(\|u_\beta\|^p_{\mathcal{D}_a^{1,p}(U,x_0)} - \|u_\beta - u\|^p_{\mathcal{D}_a^{1,p}(U,x_0)} \right) = \|u\|^p_{\mathcal{D}_a^{1,p}(U,x_0)}$*;*

(3) and

$$|\nabla u_\beta|^{p-2} \nabla u_\beta - |\nabla u_\beta - \nabla u|^{p-2} (\nabla u_\beta - \nabla u) \to |\nabla u|^{p-2} \nabla u$$

strongly in $L^{\frac{p}{p-1}}(U, |x - x_0|^{-ap})$

(4) and

$$|u_\beta|^{q-2} u_\beta - |u_\beta - u|^{q-2} (u_\beta - u) \to |u|^{q-2} u$$

strongly in $L^{\frac{q}{q-1}}(U, |x - x_0|^{-bq})$.

Proof. The first item is the conclusion of Proposition 2.1.1 whence (2) is a direct consequence of the Brézis-Lieb Lemma. Finally, (3) and (4) follows from straightforward adaptations of Alves [2, Lemma 3] and Mercuri-Willem [52, Lemma 3.2]. □

2.2 Palais-Smale Sequences and Energies of Critical Points

We now begin to treat Palais-Smale sequences for the Caffarelli-Kohn-Nirenberg problem directly. Here, we shall establish essential properties such a sequence must satisfy: e.g. boundedness in the energy space and non-negative limiting energy. In addition, we shall show that any non-trivial solution to the Caffarelli-Kohn-Nirenberg equation, in any open set, has uniformly strictly positive energy. This last detail is crucial; indeed, is it precisely this fact that ensures our iteration process in the proofs of Theorems 3.1.1-3.1.2 terminates.

Proposition 2.2.1. *A Palais-Smale sequence* (u_α) *in* $\mathcal{D}^{1,p}_a(\Omega, 0)$ *for the energy functional* J *given in* (1.1.1) *is bounded.*

Proof. We follow the argument used in Struwe [63, Lemma 2.3], which is typical for showing such sequences are bounded. To this end, let (u_α) be a Palais-Smale sequence for J. A simple calculation shows that, for some $C > 0$,

$$\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} = pJ(u_\alpha) + \frac{p}{q}\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x \leq C + \frac{p}{q}\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x$$

where we have used that $J(u_\alpha)$ is bounded in α. Next, observe that

$$\left(1 - \frac{p}{q}\right)\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x = pJ(u_\alpha) - \langle J'(u_\alpha), u_\alpha\rangle \leq C + o(1)\left\|u_\alpha\right\|_{\mathcal{D}^{1,p}_a(\Omega,0)}$$

It then follows that, for a suitable $\tilde{C} > 0$,

$$\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \leq C + \frac{p}{q}\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x \leq \tilde{C} + o(1)\left\|u_\alpha\right\|_{\mathcal{D}^{1,p}_a(\Omega,0)}$$

whence (u_α) is bounded in $\mathcal{D}^{1,p}_a(\Omega, 0)$. □

Note that, owing to the reflexivity of $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$, an immediate by product of this last proposition is the ability to pass to a subsequence (which preserves the Palais-Smale property), guaranteeing the existence of a weak limit. In fact, by our earlier discussion on $\mathcal{D}^{1,p}_a(\Omega, 0)$, we may also assume convergence almost everywhere on Ω.

Next, we show that (in the limit) Palais-Smale sequences for J are uniformly bounded from below by zero in energy. Thus, no Palais-Smale sequence may approach negative energies. More precisely, we have the following:

Lemma 2.2.2. *Let* (u_α) *be a Palais-Smale sequence for the energy functional* J *corresponding to* (A). *Then,* (u_α) *has non-negative limiting energy. More precisely,*

$$\liminf_{\alpha\to\infty} J(u_\alpha) \geq 0.$$

Proof. Since (u_α) is bounded and a Palais-Smale sequence, it is readily seen that

$$\int_\Omega |\nabla u_\alpha|^p |x|^{-ap}\,\mathrm{d}x - \int_\Omega |u_\alpha|^q |x|^{-bq}\,\mathrm{d}x = \langle J'(u_\alpha), u_\alpha\rangle = o(1)$$

as $\alpha \to \infty$. Consequently, as $\alpha \to \infty$,

$$J(u_\alpha) = \frac{\|\nabla u_\alpha\|^p_{L^p_a(\Omega,0)}}{p} - \frac{\|u_\alpha\|^q_{L^q_b(\Omega,0)}}{q} = \left(\frac{1}{p} - \frac{1}{q}\right)\|u_\alpha\|^q_{L^q_b(\Omega,0)} + o(1)$$

Passing to the limit inferior, we conclude the proof. □

Proposition 2.2.3. *Let* $C_{a,b} > 0$ *be the CKN-constant, in particular* $C_{a,b} > 0$ *is such that* (2.1.1) *holds. Let* $U \subseteq \mathbb{R}^n$ *be an open set and suppose* $u \in \mathcal{D}^{1,p}_a(U,0)$ *is a non-trivial critical point of* $J_{0,\infty}$ *in* U*, i.e.* $\left\langle J'_{0,\infty}(u), h\right\rangle = 0$ *for all* $h \in \mathcal{D}^{1,p}_a(U,0)$. *Then,*

$$0 < (C_{a,b})^{\frac{n}{p(1+a-b)}}\left(\frac{1+a-b}{n}\right) \leq J_{0,\infty}(u).$$

In particular,

1. *The constant* $(C_{a,b})^{\frac{n}{p(1+a-b)}}\left(\frac{1+a-b}{n}\right)$ *is a lower bound for the energy of non-trivial solutions to problems* (A) *and* (B).

2. *If* $a = b$*, then* $(C_{0,0})^{\frac{n}{p}}\left(\frac{1}{n}\right)$ *is a lower bound for the energy of non-trivial solutions to problems* (C) *and* (D).

Proof. Since $u \in \mathcal{D}^{1,p}_a(U,0)$ is a critical point of $J_{0,\infty}$ in U, after extending u by 0 outside of U, we have

$$0 = \left\langle J'_{0,\infty}(u), u\right\rangle = \int_{\mathbb{R}^n} |\nabla u|^p |x|^{-ap}\,\mathrm{d}x - \int_{\mathbb{R}^n} |u|^q |x|^{-bq}\,\mathrm{d}x. \tag{2.2.1}$$

Therefore,

$$J_{0,\infty}(u) = \frac{1}{p}\|\nabla u\|^p_{L^p_a(\mathbb{R}^n)} - \frac{1}{q}\|u\|^q_{L^q_b(\mathbb{R}^n)} = \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^q_{L^q_b(\mathbb{R}^n)}$$
$$= \left(\frac{1+a-b}{n}\right)\|u\|^q_{L^q_b(\mathbb{R}^n)}$$

On the other hand, by (2.1.1) and using again (2.2.1),

$$C_{a,b}\|u\|^p_{L^q_b(\mathbb{R}^n)} \le \|\nabla u\|^p_{L^p_a(\mathbb{R}^n)} = \|u\|^q_{L^q_b(\mathbb{R}^n)},$$

whence

$$0 < (C_{a,b})^{q/(q-p)} \le \|u\|^q_{L^q_b(\mathbb{R}^n)}.$$

We conclude that,

$$J_{0,\infty}(u) = \left(\frac{1+a-b}{n}\right)\|u\|^q_{L^q_b(\mathbb{R}^n)} \ge (C_{a,b})^{q/(q-p)}\left(\frac{1+a-b}{n}\right)$$
$$= (C_{a,b})^{\frac{n}{p(1+a-b)}}\left(\frac{1+a-b}{n}\right).$$

$\square$

2.3 Homogeneity and Rescaling Laws

Let $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ be given and fix an arbitrary point $x_0 \in \mathbb{R}^n$ along with some scaling factor $\lambda > 0$. We consider the following rescaling of u:

$$v(x) := \lambda^\gamma u(\lambda(x + x_0)), \tag{2.3.1}$$

where $\gamma > 0$ is the homogeneity exponent given by equation (3.5.8). With this rescaling, an easy calculation yields that

$$\|v\|_{L^q_b(\mathbb{R}^n,-x_0)} = \|u\|_{L^q_b(\mathbb{R}^n,0)} \quad \text{and} \quad \|\nabla v\|_{L^p_a(\mathbb{R}^n,-x_0)} = \|\nabla u\|_{L^p_a(\mathbb{R}^n,0)}$$

whence it follows that $v \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$.

2.3.1 Modifying the Rescaling: A New Transform

Let $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ and consider an arbitrary point $y \in \mathbb{R}^n \setminus \{0\}$ along with some $\lambda > 0$. We also fix a smooth function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} 0 \le \eta \le 1 & \\ \eta = 1 & \text{in } B(0,1/2) \\ \eta = 0 & \text{outside } B(0,1). \end{cases} \tag{2.3.2}$$

Define the transform

$$\tau_{y,\lambda} : \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0) \to \mathcal{D}^{1,p}(\mathbb{R}^n)$$

via the rule

$$\tau_{y,\lambda} u(x) = \begin{cases} \lambda^\gamma u(\lambda x + y) & \text{if } a = b = 0 \\ \left(\dfrac{\lambda}{|y|}\right)^b \lambda^\gamma u\left(\lambda x + y\right) \eta\left(\dfrac{2\lambda x}{|y|}\right) & \text{otherwise} \end{cases} \tag{2.3.3}$$

It turns out, that this transform defined is the correct tool to both test a Palais-Smale sequence (associated to the energy functional J) for *unweighted* bubbling, and for the extraction of said bubble. With this in mind, we now examine how this transformation behaves with respect to operator bounds relative to the Sobolev and Lebesgue norms.

Lemma 2.3.1. *There exists a constant $C > 0$ such that, given a point $y \in \mathbb{R}^n \setminus \{0\}$ along with some $\lambda > 0$, there holds*

$$\begin{aligned} \|\tau_{y,\lambda} u\|_{L^q(\mathbb{R}^n)} &\le C \|u\|_{L^q_b(\mathbb{R}^n,0)} \\ \|\tau_{y,\lambda} u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} &\le C \left(\frac{\lambda}{|y|}\right)^{b-a} \|u\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} \end{aligned}$$

for all $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$.

Proof. The case $a = b = 0$ follows directly from homogeneity. Otherwise, we begin with a simple observation. Suppose $x \in \mathbb{R}^n$ is such that $|2\lambda x| \le |y|$. Then, by the triangle and reverse-triangle inequalities, we then have

$$\frac{1}{2} \leq 1 - \frac{|\lambda x|}{|y|} \leq \frac{|\lambda x + y|}{|y|} \leq 1 + \frac{|\lambda x|}{|y|} \leq \frac{3}{2}.$$

Therefore, there exists a constant $C = C(ap, bq) > 0$ independent of λ, y such that

$$\left(\frac{|\lambda x + y|}{|y|}\right)^{ap} \leq C, \quad \left(\frac{|\lambda x + y|}{|y|}\right)^{bq} \leq C \tag{2.3.4}$$

for all $x \in \mathbb{R}^n$ such that $2\lambda |x| \leq |y|$. In particular, these last equations hold whenever $2\lambda x/|y|$ is in the support of η.

Given $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$, we use inequality (2.3.4) in order to bound the L^q and $\mathcal{D}^{1,p}$ norms of $v := \tau_{y,\lambda} u$. Since $0 \leq \eta \leq 1$, observe that

$$\begin{aligned}
\|v\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |v|^q \,\mathrm{d}x &\leq \left(\frac{\lambda}{|y|}\right)^{bq} \lambda^{\gamma q} \int_{2\lambda|x|\leq|y|} |u\left(\lambda x + y\right)|^q \,\mathrm{d}x \\
&= \lambda^n \int_{2\lambda|x|\leq|y|} |u\left(\lambda x + y\right)|^q \,|y|^{-bq} \,\mathrm{d}x \\
&\leq C\lambda^n \int_{2\lambda|x|\leq|y|} |u\left(\lambda x + y\right)|^q \,|\lambda x + y|^{-bq} \,\mathrm{d}x \\
&= C \int_{2|z-y|\leq|y|} |u\left(z\right)|^q \,|z|^{-bq} \,\mathrm{d}z \\
&\leq C \,\|u\|^q_{L^q_b(\mathbb{R}^n,0)}
\end{aligned}$$

which establishes the first inequality. In order to bound the $\mathcal{D}^{1,p}$-norm of v, we will consider two separate terms obtained via the product rule:

$$\begin{aligned}
\|v\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla v|^p \,\mathrm{d}x \\
&\leq 2^{p-1} \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \lambda^n \int_{\mathbb{R}^n} \left|\eta\left(\frac{2\lambda x}{|y|}\right) \nabla u(\lambda x + y)\right|^p |y|^{-ap} \,\mathrm{d}x \\
&\quad + 2^{p-1+p} \left(\frac{\lambda}{|y|}\right)^p \lambda^{np/q} \int_{\mathbb{R}^n} \left|u(\lambda x + y) \nabla\eta\left(\frac{2\lambda x}{|y|}\right)\right|^p |y|^{-bp} \,\mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned}$$

Here, we have used that $p > 1$ so that convexity implies $(\kappa + \iota)^p \leq 2^{p-1}(\kappa^p + \iota^p)$ for all $\kappa, \iota \geq 0$. We bound I_1 following the procedure used to bound the L^q-norm of v. This yields

$$I_1 \leq C 2^{p-1} \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \|u\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} .$$

For the second term, with $C_1 > 0$ such that $2^{1-2p}C_1$ is an upper bound of $|\nabla \eta|^p$, we have

$$\begin{aligned} I_2 &\leq C_1 \left(\frac{\lambda}{|y|}\right)^p \lambda^{np/q} \int_{2\lambda|x| \leq |y|} |u(\lambda x + y)|^p \, |y|^{-bp} \, \mathrm{d}x \\ &\leq C_1 \left(\frac{\lambda}{|y|}\right)^p \left(\lambda^n \int_{2\lambda|x| \leq |y|} |u(\lambda x + y)|^q \, |y|^{-bq} \, \mathrm{d}x\right)^{p/q} \left(\int_{2\lambda|x|\leq|y|} 1 \mathrm{d}x\right)^{p(1+a-b)/n} \end{aligned}$$

by Hölder's inequality (with Hölder exponent q/p and Hölder-conjugate exponent $n/[p(1+a-b)]$). Then, up to a modification of the constant $C > 0$, we see that

$$\begin{aligned} I_2 &\leq C \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \left(\lambda^n \int_{2\lambda|x|\leq|y|} |u(\lambda x + y)|^q \, |\lambda x + y|^{-bq} \, \mathrm{d}x\right)^{p/q} \\ &= C \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \left(\int_{2|z-y|\leq|y|} |u(z)|^q \, |z|^{-bq} \, \mathrm{d}x\right)^{p/q} \\ &\leq C \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \|u\|^p_{L^q_b(\mathbb{R}^n,0)} . \end{aligned}$$

Finally, by the CKN inequality (CKN), up to another modification of the constant $C > 0$, we deduce that

$$I_2 \leq C \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \|u\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} .$$

Combining this with our inequality for I_1, we deduce that (possibly adjusting $C > 0$ again)

$$\|v\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq C \left(\frac{\lambda}{|y|}\right)^{p(b-a)} \|u\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)}$$

which completes our proof since C is independent of u, y, λ. □

Remark 2.3.1. Inspecting the proof of Lemma 2.3.1, it is easy to see that we may instead bound the norms on an arbitrary measurable set $U \subseteq \mathbb{R}^n$. Indeed, the very same argument shows that the same constant $C > 0$ (which is independent of y, λ, U) satisfies

$$\begin{aligned} \|\tau_{y,\lambda} u\|_{L^q(U)} &\leq C \|u\|_{L^q_b(\lambda U + y, 0)} \\ \|\nabla [\tau_{y,\lambda} u]\|_{L^p(U)} &\leq C \left(\frac{\lambda}{|y|}\right)^{b-a} \left(\|\nabla u\|_{L^p_a(\lambda U + y, 0)} + \|u\|_{L^q_b(\lambda U + y, 0)}\right) \end{aligned} \tag{2.3.5}$$

for all $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$.

In line with what we have done thus far, we shall also require a suitable "reverse" transform which will serve as a pseudoinverse for $\tau_{y,\lambda}$. Formally, we define

$$\tau^-_{y,\lambda} : \mathcal{D}^{1,p}(\mathbb{R}^n) \to \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$$

via the rule

$$\tau^-_{y,\lambda} v(z) = \begin{cases} \lambda^{-\gamma} v\left(\dfrac{z-y}{\lambda}\right) & \text{if } a = b = 0 \\ \left(\dfrac{\lambda}{|y|}\right)^{-b} \lambda^{-\gamma} v\left(\dfrac{z-y}{\lambda}\right) \eta\left(\dfrac{2(z-y)}{|y|}\right) & \text{otherwise.} \end{cases} \tag{2.3.6}$$

Obviously, by inspection, it is easy to check that $\tau^-_{y,\lambda}$ is the proper inverse of $\tau_{y,\lambda}$ when $a = b = 0$. With this in mind, let us now establish a partial analog of Lemma 2.3.1, thereby providing operator bounds related to this pseudo-inversion mapping.

Lemma 2.3.2. *There exists a constant $C > 0$ such that, given a point $y \in \mathbb{R}^n \setminus \{0\}$ along with some $\lambda > 0$, there holds*

$$\begin{aligned} \left\|\tau^-_{y,\lambda} v\right\|_{L^q_b(\mathbb{R}^n, 0)} &\leq C \|v\|_{L^q(\mathbb{R}^n)} \\ \left\|\tau^-_{y,\lambda} v\right\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)} &\leq C \left(\frac{|y|}{\lambda}\right)^{p(b-a)} \|v\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \end{aligned}$$

for all $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$.

Proof. Once again, the case $a = b = 0$ is trivial since it follows from homogeneity. Otherwise, as in the proof of Lemma 2.3.1 with equation (2.3.4), we see that there exists a constant $C > 0$ independent of y, λ such that

$$\left(\frac{|\lambda x + y|}{|y|}\right)^{-ap} \leq C, \quad \left(\frac{|\lambda x + y|}{|y|}\right)^{-bq} \leq C \tag{2.3.7}$$

for all $x \in \mathbb{R}^n$ satisfying $2\lambda |x| \leq |y|$. In particular, (2.3.7) is valid for all x such that $2\lambda x / |y|$ is in the support of η.

For an arbitrary $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, set $u := \tau_{y,\lambda}^{-} v$. Using that $0 \leq \eta \leq 1$ we see that

$$\begin{aligned}
\|u\|^q_{L^q_b(\mathbb{R}^n,0)} &= \int_{\mathbb{R}^n} |u(z)|^q \, |z|^{-bq} \, \mathrm{d}z \\
&\leq \left(\frac{\lambda}{|y|}\right)^{-bq} \lambda^{-\gamma q} \int_{2|z-y| \leq |y|} \left| v\left(\frac{z-y}{\lambda}\right)\right|^q |z|^{-bq} \, \mathrm{d}z \\
&= \int_{2\lambda |x| \leq |y|} |v(x)|^q \left(\frac{|\lambda x + y|}{|y|}\right)^{-bq} \mathrm{d}x \\
&\leq C \, \|v\|^q_{L^q(\mathbb{R}^n)} .
\end{aligned}$$

Next, in order to bound the $\mathcal{D}^{1,p}_a$ norm of u, we once again apply the product rule to obtain two terms:

$$\begin{aligned}
\|u\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} &= \int_{\mathbb{R}^n} |\nabla u(z)|^p \, |z|^{-ap} \, \mathrm{d}z \\
&\leq 2^{p-1} \left(\frac{\lambda}{|y|}\right)^{-bp} \lambda^{-n+ap} \int_{\mathbb{R}^n} \left| \eta\left(\frac{2(z-y)}{|y|}\right) \nabla v\left(\frac{z-y}{\lambda}\right)\right|^p |z|^{-ap} \, \mathrm{d}z \\
&\quad + 2^{p-1+p} \left(\frac{\lambda}{|y|}\right)^{-bp} \lambda^{-\gamma p} \, |y|^{-p} \int_{\mathbb{R}^n} \left| v\left(\frac{z-y}{\lambda}\right) \nabla \eta\left(\frac{2(z-y)}{|y|}\right)\right|^p |z|^{-ap} \, \mathrm{d}z \\
&=: I_1 + I_2
\end{aligned}$$

where we have used that $(\kappa + \iota)^p \leq 2^{p-1} \left(\kappa^p + \iota^p\right)$ for all $\kappa, \iota \geq 0$. Then, after a change of variables, observe that

$$\begin{aligned}
I_1 &\leq 2^{p-1} \left(\frac{\lambda}{|y|}\right)^{-bp} \lambda^{ap} \int_{2\lambda|x| \leq |y|} |\nabla v(x)|^p \, |\lambda x + y|^{-ap} \, \mathrm{d}x \\
&\leq C 2^{p-1} \left(\frac{|y|}{\lambda}\right)^{p(b-a)} \|v\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} .
\end{aligned}$$

Similarly, with $C_1 > 0$ such that $2^{1-2p}C_1$ is an upper bound of $|\nabla \eta|^p$ we see that

$$\begin{aligned} I_2 &\leq C_1 \left(\frac{\lambda}{|y|}\right)^{-bp} \lambda^{ap+p} |y|^{-p} \int_{2\lambda|x|\leq|y|} |v(x)|^p |\lambda x + y|^{-ap} \,\mathrm{d}x \\ &\leq CC_1 \left(\frac{\lambda}{|y|}\right)^{p(1+a-b)} \int_{2\lambda|x|\leq|y|} |v(x)|^p \,\mathrm{d}x. \end{aligned}$$

Then, by Hölder's inequality (with exponent p^*/p and Hölder-conjugate exponent n/p),

$$\begin{aligned} I_2 &\leq CC_1 2^p \left(\frac{\lambda}{|y|}\right)^{p(1+a-b)} \|v\|^p_{L^{p^*}(\mathbb{R}^n)} \left(\int_{2\lambda|x|\leq|y|} 1\right)^{p/n} \\ &\leq C \left(\frac{|y|}{\lambda}\right)^{p(b-a)} \|v\|^p_{L^{p^*}(\mathbb{R}^n)} \end{aligned}$$

up to an adjustment of the constant C in the second inequality. We then conclude our proof via an application of the Gagliardo-Nirenberg-Sobolev inequality (i.e. the CKN inequality (CKN) with $a = b = 0$) and combining our bounds for I_1 and I_2. □

Remark 2.3.2. Let now $U \subseteq \mathbb{R}^n$ be Lebesgue measurable. As was noted for the forward transform, we observe that an identical argument exhibits the existence of a constant $C > 0$ (independent of y, λ, U) such that

$$\begin{aligned} \left\|\tau^-_{y,\lambda} v\right\|_{L^q_b(U,0)} &\leq C \|v\|_{L^q([U-y]/\lambda)} \\ \left\|\nabla \left[\tau^-_{y,\lambda} v\right]\right\|_{L^p_a(U,0)} &\leq C \left(\frac{|y|}{\lambda}\right)^{p(b-a)} \left(\|v\|_{L^q([U-y]/\lambda)} + \|\nabla v\|_{L^p([U-y]/\lambda)}\right) \end{aligned} \tag{2.3.8}$$

for all $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$.

Transformations and the L^q-norm

As was shown, the transform $\tau_{y,\lambda}$ is particularly well behaved with respect to the L^q-norm. It turns out that when $|y|/\lambda$ is large, the L^q-norm of the transform remains *almost* unchanged on a set where η acts as the identity. More precisely, we have the following result:

Lemma 2.3.3. *There exists a constant $C > 0$ such that, for any $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ and $y \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$ such that $|y| \geq 4\lambda$, the remainder R in*

$$\|\tau_{y,\lambda} u\|^q_{L^q(B(0,1))} = \|u\|^q_{L^q_b(B(y,\lambda),0)} + R$$

satisfies

$$|R| \leq C \frac{\lambda}{|y|} \|u\|^q_{L^q_b(B(y,\lambda),0)}$$

Before presenting the proof, we recall that Lemma 2.3.1 and 2.3.2 relied on η in order to restrict our support; then, on this set, we utilized uniform bounds. Estimates of this form will be of repeated use and are thus presented independently as follows:

Lemma 2.3.4. *There exists a constant $C > 0$ such that*

(a) Given $y \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$,

$$\frac{1}{C} \leq \frac{|\lambda x + y|}{|y|} \leq C \quad \textit{for all } x \in \mathbb{R}^n \textit{ such that } |x| \leq \frac{|y|}{2\lambda}.$$

(b) Given $y \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$ and any bounded set K such that $\rho := \sup_{x \in K} |x|$ satisfies $4\lambda\rho \leq |y|$, there holds

$$\frac{\left||y|^{-ap} - |\lambda x + y|^{-ap}\right|}{|y|^{-ap}} \leq C\rho \frac{\lambda}{|y|}, \qquad \frac{\left||y|^{-bq} - |\lambda x + y|^{-bq}\right|}{|y|^{-bq}} \leq C\rho \frac{\lambda}{|y|}$$

for all $x \in K$.

Proof. The first inequality follows from the reverse and usual triangle inequalities. Next, let y, λ, K, ρ be as in (b). Given an arbitrary $x \in K$ write $z := \lambda x + y$ and observe that

$$|z| \leq |y| + \lambda |x| \leq \left(1 + \frac{\lambda\rho}{|y|}\right) |y| .$$

Using also the reverse triangle inequality, we see that

$$(1 - \varepsilon) |y| \leq |z| \leq (1 + \varepsilon) |y|$$

where $\varepsilon := \lambda\rho/|y| \le 1/4$. From this, we further deduce that

$$\left||y|^{-ap} - |z|^{-ap}\right| \le \left|1 - (1+\varepsilon_{\pm})^{-ap}\right| |y|^{-ap}$$

where $\varepsilon_{\pm}$ is either ε or $-\varepsilon$. It then follows from the Mean Value Theorem that

$$\left||y|^{-ap} - |z|^{-ap}\right| \le |ap|\,(1+\delta)^{-ap-1}\,\varepsilon\,|y|^{-ap}$$

for some δ in $[-\varepsilon, \varepsilon]$. From this last equation and recalling that $|\delta| \le \varepsilon \le 1/4$, we see that there exists a constant $C > 0$ independent of y, λ such that

$$\frac{\left||y|^{-ap} - |z|^{-ap}\right|}{|y|^{-ap}} \le C\varepsilon, \qquad \forall z \in \lambda K + y.$$

Analogously, modifying $C > 0$ if necessary, we may establish that

$$\frac{\left||y|^{-bq} - |z|^{-bq}\right|}{|y|^{-bq}} \le C\varepsilon, \qquad \forall z \in \lambda K + y.$$

$\square$

With this, we are ready to prove Lemma 2.3.3

Proof of Lemma 2.3.3. Let $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$, $y \in \mathbb{R}^n \setminus \{0\}$, and $\lambda > 0$ such that $|y| \ge 4\lambda$. Properties of η imply that

$$\begin{aligned}\int_{B(0,1)} |\tau_{y,\lambda} u|^q \,\mathrm{d}x &= \lambda^n \int_{B(0,1)} |u(\lambda x + y)|^q\, |y|^{-bq} \,\mathrm{d}x \\ &= \int_{B(y,\lambda)} |u(z)|^q\, |y|^{-bq}\,\mathrm{d}z,\end{aligned}$$

where we have used a change of variables in this last equality. Therefore,

$$\|\tau_{y,\lambda} u\|^q_{L^q(B(0,1))} - \|u\|^q_{L^q_b(B(y,\lambda),0)} = \int_{B(y,\lambda)} |u(z)|^q \left(|y|^{-bq} - |z|^{-bq}\right) \mathrm{d}z.$$

Then, using both parts of Lemma 2.3.4, we infer that there is a constant $C > 0$

independent of y, λ, u such that

$$\left| \|\tau_{y,\lambda} u\|^q_{L^q(B(0,1))} - \|u\|^q_{L^q_b(B(y,\lambda),0)} \right| \leq C \frac{\lambda}{|y|} \int_{B(y,\lambda)} |u(z)|^q \, |z|^{-bq} \, \mathrm{d}z$$

which concludes our proof. □

2.3.2 Locality

A useful property of the transforms we introduced is, when $a = b$, the ability to restrict domains. More specifically, we have the following technical result.

Lemma 2.3.5. *Given $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, if $a = b$ then*

$$\left\|\tau^-_{y,\lambda} v\right\|_{L^q_b(\mathbb{R}^n \setminus B(y,|y|/4))} \to 0, \quad \textit{and} \quad \left\|\nabla \left[\tau^-_{y,\lambda} v\right]\right\|_{L^p_a(\mathbb{R}^n \setminus B(y,|y|/4))} \to 0$$

as $|y|/\lambda \to \infty$.

Proof. This result follows almost immediately from Remark 2.3.2. Indeed, there is a constant $C > 0$ independent of λ, y, v such that

$$\left\|\tau^-_{y,\lambda} v\right\|_{L^q_b(\mathbb{R}^n \setminus B(y,|y|/4))} \leq C \|v\|_{L^q\left(\mathbb{R}^n \setminus B\left(0,\frac{|y|}{4\lambda}\right)\right)}.$$

Now, since $a = b$ we note that the Sobolev exponent is

$$p^* = \frac{np}{n-p} = q.$$

Therefore, $v \in L^q(\mathbb{R}^n)$ and, as $|y|/\lambda \to \infty$, the dominated convergence theorem implies that

$$\left\|\tau^-_{y,\lambda} v\right\|_{L^q_b(\mathbb{R}^n \setminus B(y,|y|/4))} \leq C \|v\|_{L^q\left(\mathbb{R}^n \setminus B\left(0,\frac{|y|}{4\lambda}\right)\right)} \to 0.$$

Similarly, since $a = b$, Remark 2.3.2 allows us to write

$$\left\|\nabla \left[\tau^-_{y,\lambda} v\right]\right\|_{L^p_a(\mathbb{R}^n \setminus B(y,|y|/4))} \leq C \left(\|v\|_{L^q\left(\mathbb{R}^n \setminus B\left(0,\frac{|y|}{4\lambda}\right)\right)} + \|\nabla v\|_{L^p\left(\mathbb{R}^n \setminus B\left(0,\frac{|y|}{4\lambda}\right)\right)} \right)$$

for some $C > 0$ independent of y, λ, v. Once again, the dominated convergence theorem implies that the right-hand side tends to 0 as $|y|/\lambda \to \infty$. $\square$

2.4 Convergence of Domains

In the discussion surrounding our preliminary statement of Theorem 1.1.2, we noted the possibility that a bubble solving the unweighted half-space problem (D) arises. It is important to realize that bubbles supported in a half-space is not at all a new phenomenon, and that what is special in our case is the presence of both weighted and unweighted limiting problems. In fact, although their final statements

As in the previous sections, we continue to assume that $\Omega \subset \mathbb{R}^n$ denotes a bounded domain containing the origin. Throughout this section, we fix sequences (y_α) in $\overline{\Omega}$ and λ_α in $(0, \infty)$ satisfying some combination of the following properties:

(P1) $\lambda_\alpha \to 0$;

(P2) The ratio converges in the sense that, for some $x_0 \in \mathbb{R}^n$,

$$\frac{y_\alpha}{\lambda_\alpha} \to x_0.$$

(P3) The ratio diverges:

$$\frac{|y_\alpha|}{\lambda_\alpha} \to \infty,$$

(P4) The sequence (y_α) approaches the boundary at rate no less than the rescaling factor; i.e. the following ratio is bounded:

$$\frac{\operatorname{dist}(y_\alpha, \partial\Omega)}{\lambda_\alpha}.$$

(P5) The sequence (y_α) approaches the boundary at a rate slower than the rescaling factor; i.e.

$$\frac{\operatorname{dist}(y_\alpha, \partial\Omega)}{\lambda_\alpha} \to \infty.$$

The goal of this section is to study the limiting behaviour of specific sequences of sets. First, we describe the sense in which we consider the convergence of sets.

Definition 2.4.1. Let (U_α) be a sequence of subsets of $\mathbb{R}^n$. We say that (U_α) converges to $U \subseteq \mathbb{R}^n$ and write $U_\alpha \to U$ provided

(a) Any compact subset of U is contained in U_α for all α large.

(b) Any compact subset of $\left(\overline{U}\right)^\complement$ is contained in $\left(\overline{U_\alpha}\right)^\complement \subseteq U_\alpha^\complement$ for all α large.

Now, suppose condition (P2) holds. Then we consider the following deformations of Ω:

$$\Omega_\alpha = \frac{\Omega}{\lambda_\alpha} - x_0.$$

Clearly, these sets are approximations of $(\Omega - y_\alpha)/\lambda_\alpha$. Furthermore, we have the following result:

Proposition 2.4.1. *Suppose the sequence (λ_α) converges. Then, we have $\Omega_\alpha \to \Omega_\infty$ where*

$$\Omega_\infty = \begin{cases} \mathbb{R}^n & \text{if condition (P1) holds, i.e. } \lambda_\alpha \to 0 \\ \dfrac{\Omega}{\lambda} - x_0 & \text{if } \lambda_\alpha \to \lambda > 0. \end{cases}$$

Proof. We first treat the case where $\lambda_\alpha \to 0$. Since Ω is an open set containing the origin, we have $B(0,\delta) \subseteq \Omega$ for some $\delta > 0$. It readily follows that

$$B\left(-x_0, \frac{\delta}{\lambda_\alpha}\right) \subseteq \Omega_\alpha$$

for every index α. Then, for an arbitrary compact set K, using $\lambda_\alpha \to 0$ we see that

$$K \subseteq B\left(-x_0, \frac{\delta}{\lambda_\alpha}\right) \subseteq \Omega_\alpha$$

for all α large. Hence, $\Omega_\alpha \to \mathbb{R}^n$.

Let us now assume that we are in the case $\lambda_\alpha \to \lambda > 0$ and fix a compact set $K \subseteq \Omega_\infty$. Then, $\lambda(K + x_0)$ is a compact subset of Ω. Since Ω is open, the Lebesgue Number Lemma guarantees the existence of $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq \Omega$ for all $x \in \lambda(K + x_0)$. Put otherwise, for all $z \in K$ there holds

$$B\left(\frac{\lambda}{\lambda_\alpha}(z + x_0) - x_0, \frac{\varepsilon}{\lambda_\alpha}\right) \subseteq \Omega_\alpha.$$

On the other hand, since $K + x_0$ is compact, it is bounded by some $M > 0$. Then, for any $z \in K$, if α is such that

$$|\lambda_\alpha - \lambda| < \frac{\varepsilon}{M}$$

then

$$\left| z - \left(\frac{\lambda}{\lambda_\alpha}(z + x_0) - x_0 \right) \right| = \frac{1}{\lambda_\alpha} |\lambda - \lambda_\alpha| \, |z + x_0| < \frac{\varepsilon}{\lambda_\alpha}.$$

Since $\lambda_\alpha \to \lambda > 0$, it follows that for all α sufficiently large we have

$$K \subseteq B\left(\frac{\lambda}{\lambda_\alpha}(z + x_0) - x_0, \frac{\varepsilon}{\lambda_\alpha} \right) \subseteq \Omega_\alpha.$$

Finally, given a compact set $K \subseteq \left(\overline{\Omega_\infty}\right)^{\complement}$, the same argument shows that $K \subseteq \overline{\Omega_\alpha}^{\complement}$ for all α large. We once again conclude that $\Omega_\alpha \to \Omega_\infty$. □

Next, suppose condition (P3) holds. Then we will be interested in the following deformations of Ω:

$$\Omega_\alpha = \frac{\Omega - y_\alpha}{\lambda_\alpha}.$$

In this case, if condition (P1) holds then we have the following result:

Proposition 2.4.2. *Suppose Ω has C^1 boundary. If (P1) holds, i.e. $\lambda_\alpha \to 0$, then, up to a subsequence, $\Omega_\alpha := (\Omega - y_\alpha)/\lambda_\alpha$ tends to a set $\Omega_\infty \subseteq \mathbb{R}^n$.*

1. *If the distance between y_α and the boundary tends to 0 no slower than (λ_α), i.e. if condition (P4) holds, then Ω_∞ is a half-space.*

2. *On the other hand, if condition (P5) holds then $\Omega_\infty = \mathbb{R}^n$.*

The proof is a direct adaption of El-Hamidi-Vétois [33, Proposition 2.4], describing the convergence of domains. We have chosen to include the precise statement to ensure a rigorous treatment of the half-space case.

Next, having identified the limiting domains we might encounter when extracting our bubbling information, we verify that the appropriate limit solves the expected problem. Intuitively, this is quite clear but nonetheless requires a proof.

Lemma 2.4.3. *Suppose $\Omega_\alpha \to \Omega_\infty$, with Ω_∞ being either $\mathbb{R}^n$ or a half space. Let (v_α) in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ be a sequence converging to some $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ pointwise almost everywhere. If $v_\alpha \in \mathcal{D}^{1,p}(\Omega_\alpha)$ for each α then $v \in \mathcal{D}^{1,p}(\Omega_\infty)$.*

Proof. If $\Omega_\infty = \mathbb{R}^n$ there is nothing to show. Otherwise, up to a translation, there exists a unit vector ν such that

$$\Omega_\infty = \{x \in \mathbb{R}^n : x \cdot \nu > 0\}.$$

Now, observe that $J_\varepsilon(x) := v(x - \varepsilon\nu)$ converges to v in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ as $\varepsilon \searrow 0$ by standard results. Therefore, if we can show that $J_\varepsilon \in \mathcal{D}^{1,p}(\Omega_\infty)$ for each $\varepsilon > 0$ then we must have $v \in \mathcal{D}^{1,p}(\Omega_\infty)$ which completes the proof.

To this end, we first show that v vanishes outside of $\overline{\Omega_\infty}$ by showing that v vanishes on every compact set K such that $K \cap \overline{\Omega_\infty} = \varnothing$. Indeed, if K is compact and does not intersect $\overline{\Omega_\infty}$ then, since $\Omega_\alpha \to \Omega_\infty$, we see that $K \cap \overline{\Omega_\alpha} = \varnothing$ for all α large. Thus, v_α vanishes on K for each α large. Since $v_\alpha \to v$ pointwise almost everywhere, it follows that v vanishes on K.

Having shown that v is supported on $\overline{\Omega_\infty}$, it readily follows that, every J_ε is supported on

$$\{x \in \mathbb{R}^n : x \cdot \nu \geq \varepsilon\}.$$

Now, in order to prove that $J_\varepsilon \in \mathcal{D}^{1,p}(\Omega_\infty)$ we show that, for any $\delta > 0$, we may find a function in $C_c^\infty(\Omega_\infty)$ whose $\mathcal{D}^{1,p}$-distance to J_ε is less than δ. With this in mind, let $\delta > 0$ be given and set $\zeta \in C_c^\infty(\mathbb{R}^n; [0,1])$ to be a smooth function of compact support with $\zeta \equiv 1$ on $B(0,1)$. Then, for any $h_1 > 0$, define

$$\zeta_{h_1}(x) := \zeta(h_1 x)$$

so that $J_\varepsilon \zeta_{h_1} \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. By the product rule and using that $(\iota+\kappa)^p \leq 2^{p-1}(\iota^p+\kappa^p)$ for all $\iota, \kappa \geq 0$,

$$\|J_\varepsilon - \zeta_{h_1} J_\varepsilon\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \leq 2^{p-1} \int_{|x| \geq 1/h_1} |\nabla J_\varepsilon|^p \, \mathrm{d}x + 2^{p-1} \int_{|x| \geq 1/h_1} |J_\varepsilon \nabla \zeta_{h_1}|^p \, \mathrm{d}x. \tag{2.4.1}$$

The first term on the right-hand side tends to 0 as $h_1 \searrow 0$. As for the second term, by Hölder's inequality with exponent q/p and conjugate exponent $n/[p(1 + a - b)]$

we may write

$$\begin{aligned}\int_{|x|\geq h_1} |J_\varepsilon \nabla \zeta_{h_1}|^p \,\mathrm{d}x &\leq \left(\int_{|x|\geq 1/h_1} |J_\varepsilon|^q \,\mathrm{d}x\right)^{p/q} \left(\int_{|x|\geq 1/h_1} |\nabla \zeta_{h_1}|^{\frac{n}{1+a-b}} \,\mathrm{d}x\right)^{p(1+a-b)/n} \\ &= \left(\int_{|x|\geq 1/h_1} |J_\varepsilon|^q \,\mathrm{d}x\right)^{p/q} \left(h_1^{\frac{n}{1+a-b}-n} \int_{|y|\geq 1} |\nabla \zeta|^{\frac{n}{1+a-b}} \,\mathrm{d}y\right)^{p(1+a-b)/n}\end{aligned}$$

by a change of variables. Thus, we see that the right-hand side tends to 0 as $h_1 \searrow 0$. Assembling, we deduce that both terms on the right-hand side of equation (2.4.1) tend to 0 as $h_1 \searrow 0$. Thus, we may select $h_1 > 0$ sufficiently small so that

$$\|J_\varepsilon - \zeta_{h_1} J_\varepsilon\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} < \frac{\delta}{2}.$$

Now, for any $0 < h \ll \varepsilon$, observe that the mollification $(\zeta_{h_1} J_\varepsilon)_h$ is in $C_c^\infty(\Omega_\infty)$. Then, by standard mollification results, we may select $h > 0$ sufficiently small so that

$$\|\zeta_{h_1} J_\varepsilon - (\zeta_{h_1} J_\varepsilon)_h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} < \frac{\delta}{2}.$$

By the triangle inequality, $\|J_\varepsilon - (\zeta_{h_1} J_\varepsilon)_h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} < \delta$ and we may conclude that $J_\varepsilon \in \mathcal{D}^{1,p}(\Omega_\infty)$, as desired. □

2.5 Energy and the New Transforms

We continue to assume that $\Omega \subset \mathbb{R}^n$ denotes a bounded domain containing the origin and fix sequences (y_α) in $\overline{\Omega}$ and λ_α in $(0, \infty)$. As in the previous section, the sequences $(y_\alpha), (\lambda_\alpha)$ will satisfy some combination of properties (P1)-(P5).

In this section, we investigate how the new transforms $\tau_{y,\lambda}, \tau_{y,\lambda}^-$ interact with the energy functionals J and J_∞. First, observe that if $a = b = 0$, a change of variables shows that

$$\langle J'_\infty(\tau_{y,\lambda} u), h\rangle = \left\langle J'(u), \tau_{y,\lambda}^- h\right\rangle.$$

for any $h \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. More generally, when $a = b$, the difference between terms on either side of the equality will be negligible provided $|y|/\lambda$ is large. This is made precise via the following result:

Lemma 2.5.1. *If $a = b$ (so $q = p^*$), there exists a constant $C > 0$ such that the remainder*

$$R := \langle J'_\infty(\tau_{y,\lambda}u), h\rangle - \left\langle J'(u), \tau^-_{y,\lambda}h\right\rangle$$

satisfies

$$|R| \leq C\rho\frac{\lambda}{|y|}\left(\|u\|^{p-1}_{\mathcal{D}^{1,p}_a(\Omega,0)}\|h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} + \|u\|^{q-1}_{L^q_b(\Omega,0)}\|h\|_{L^q(\mathbb{R}^n)}\right)$$

for any $u \in \mathcal{D}^{1,p}_a(\Omega,0)$, $y \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$ and $h \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ with support K satisfying

$$\rho := \sup_{x\in K}|x| \leq \frac{|y|}{4\lambda}, \qquad K \subseteq \frac{\Omega - y}{\lambda}. \tag{2.5.1}$$

In particular, suppose $\Omega_\infty \subseteq \mathbb{R}^n$ is such that $\frac{\Omega - y_\alpha}{\lambda_\alpha} =: \Omega_\alpha \to \Omega_\infty$. Let (u_α) be a bounded sequence in $\mathcal{D}^{1,p}_a(\Omega,0)$ and assume property (P3) holds (i.e. $|y_\alpha|/\lambda_\alpha \to \infty$). Then, given a fixed compact set $K \subseteq \Omega_\infty$,

$$\langle J'_\infty(\tau_{y_\alpha,\lambda_\alpha}u_\alpha), h\rangle - \left\langle J'(u_\alpha), \tau^-_{y_\alpha,\lambda_\alpha}h\right\rangle \to 0$$

uniformly over $h \in \{\mathcal{D}^{1,p}(\mathbb{R}^n) : \operatorname{supp}(h) \subseteq K\}$ as $\alpha \to \infty$.

Proof. Let $u \in \mathcal{D}^{1,p}_a(\Omega,0)$, $y \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$ and suppose $h \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ has support K satisfying (2.5.1). First, notice that the inclusion $\lambda K + y \subseteq \Omega$ implies that

$$R = \langle J'_\infty(\tau_{y,\lambda}u), h\rangle - \left\langle J'(u), \tau^-_{y,\lambda}h\right\rangle$$

is well-defined since $\tau^-_{y,\lambda}h \in \mathcal{D}^{1,p}_a(\Omega,0)$. Moreover, because $\rho \leq |y|/(4\lambda)$, any $x \in K = \operatorname{supp}(h)$ satisfies

$$\left|\frac{2\lambda x}{|y|}\right| \leq \frac{1}{2} \implies \eta\left(\frac{2\lambda x}{|y|}\right) = 1 \tag{2.5.2}$$

Using this , we see that

$$\begin{aligned}\langle J'_\infty(\tau_{y,\lambda}u), h\rangle = {}& \left(\frac{\lambda}{|y|}\right)^{b(p-1)}\lambda^{(\gamma+1)(p-1)}\int_{\mathbb{R}^n}|\nabla u(\lambda x+y)|^{p-2}\left(\nabla u(\lambda x+y), \nabla h(x)\right)\mathrm{d}x \\ & - \left(\frac{\lambda}{|y|}\right)^{b(q-1)}\lambda^{\gamma(q-1)}\int_{\mathbb{R}^n}|u(\lambda x+y)|^{q-2}u(\lambda x+y)h(x)\mathrm{d}x\end{aligned}$$

Then, by a change of variables,

$$\begin{aligned}\langle J'_\infty(\tau_{y,\lambda}u), h\rangle = &\left(\frac{\lambda}{|y|}\right)^{b(p-1)} \lambda^{(\gamma+1)(p-1)-n} \int_{\mathbb{R}^n} |\nabla u(z)|^{p-2} \left(\nabla u(z), \nabla h\left(\frac{z-y}{\lambda}\right)\right) \mathrm{d}z \\ &- \left(\frac{\lambda}{|y|}\right)^{b(q-1)} \lambda^{\gamma(q-1)-n} \int_{\mathbb{R}^n} |u(z)|^{q-2} u(z) h\left(\frac{z-y}{\lambda}\right) \mathrm{d}z.\end{aligned}$$

Using equation (2.5.2), but with $x = (z-y)/\lambda$, yields that

$$\begin{aligned}\langle J'_\infty(\tau_{y,\lambda}u), h\rangle = &\left(\frac{\lambda}{|y|}\right)^{bp} \lambda^{(\gamma+1)p-n} \int_{\mathbb{R}^n} |\nabla u(z)|^{p-2} \left(\nabla u(z), \nabla \left[\tau^-_{y,\lambda}h\right](z)\right) \mathrm{d}z \\ &- \left(\frac{\lambda}{|y|}\right)^{bq} \lambda^{\gamma q-n} \int_{\mathbb{R}^n} |u(z)|^{q-2} u(z) \left[\tau^-_{y,\lambda}h\right](z) \mathrm{d}z\end{aligned}$$

where u is extended by 0 outside of Ω. Simplifying and re-arranging, we obtain

$$\begin{aligned}\langle J'_\infty(\tau_{y,\lambda}u), h\rangle = &\left(\frac{\lambda}{|y|}\right)^{p(b-a)} \int_{\mathbb{R}^n} |\nabla u(z)|^{p-2} \left(\nabla u(z), \nabla \left[\tau^-_{y,\lambda}h\right](z)\right) |y|^{-ap} \mathrm{d}z \\ &- \int_{\mathbb{R}^n} |u(z)|^{q-2} u(z) \left[\tau^-_{y,\lambda}h\right](z) |y|^{-bq} \mathrm{d}z\end{aligned}$$

With this, observe that since $a = b$

$$\begin{aligned}R := &\langle J'_\infty(\tau_{y,\lambda}u), h\rangle - \left\langle J'(u), \tau^-_{y,\lambda}h\right\rangle \\ = &\int_{\mathbb{R}^n} |\nabla u(z)|^{p-2} \left(\nabla u(z), \nabla \left[\tau^-_{y,\lambda}h\right](z)\right) \left(|y|^{-ap} - |z|^{-ap}\right) \mathrm{d}z \\ &- \int_{\mathbb{R}^n} |u(z)|^{q-2} u(z) \left[\tau^-_{y,\lambda}h\right](z) \left(|y|^{-bq} - |z|^{-bq}\right) \mathrm{d}z.\end{aligned}$$

Then, using Lemma 2.3.4, we see that there is a constant $C > 0$ independent of y, λ, u, K such that

$$\begin{aligned}|R| \leq &C\varepsilon \int_{\mathbb{R}^n} |\nabla u(z)|^{p-1} \left|\nabla \left[\tau^-_{y,\lambda}h\right](z)\right| |z|^{-ap} \mathrm{d}z \\ &+ C\varepsilon \int_{\mathbb{R}^n} |u(z)|^{q-1} \left|\left[\tau^-_{y,\lambda}h\right](z)\right| |z|^{-bq} \mathrm{d}z \\ \leq &C\varepsilon \left(\|u\|^{p-1}_{\mathcal{D}^{1,p}_a(\Omega,0)} \left\|\tau^-_{y,\lambda}h\right\|_{\mathcal{D}^{1,p}_a(\Omega,0)} + \|u\|^{q-1}_{L^q_b(\Omega,0)} \left\|\tau^-_{y,\lambda}h\right\|_{L^q_b(\Omega,0)}\right)\end{aligned}$$

where $\varepsilon := \lambda\rho/|y|$ and the second inequalities follows from Hölder's inequality. We note that, for the last line, we have implicitly used that u and $\tau_{y,\lambda}^{-}h$ are in $\mathcal{D}_a^{1,p}(\Omega,0)$ and extended by 0 to the rest of $\mathbb{R}^n$. Then, by Lemma 2.3.2, with a possible modification of the constant $C > 0$ we deduce that

$$|R| \leq C\varepsilon \left(\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)}^{p-1} \|h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} + \|u\|_{L_b^q(\Omega,0)}^{q-1} \|h\|_{L^q(\mathbb{R}^n)} \right)$$

where we have once again used that $a = b$. Finally,the second part of the Lemma follows from this last inequality and the Gagliardo-Nirenberg-Sobolev inequality. □

In this last lemma, we studied how the composition $J_\infty' \circ \tau_{y,\lambda}$ behaves by testing against a fixed function h. In a similar vein, with the aims of recovering a (PS)-sequence after subtracting bubbles, let us now analyze the behaviour of J_∞' when tested against transformed functions:

Lemma 2.5.2. *Suppose $\Omega_\alpha = (\Omega - y_\alpha)/\lambda_\alpha$ tends to Ω_∞, with Ω_∞ being either a half-space or all of $\mathbb{R}^n$. If $a = b$, (P3) holds and $v \in \mathcal{D}^{1,p}(\Omega_\infty)$ is a critical point of J_∞ in Ω_∞, i.e. $\langle J_\infty'(v), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}(\Omega_\infty)$, then*

$$\langle J_\infty'(v), \tau_{y_\alpha,\lambda_\alpha} g\rangle = o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right) \quad \textit{as} \quad \alpha \to \infty.$$

over $g \in \mathcal{D}_a^{1,p}(\Omega, 0)$.

Proof. By way of contradiction, suppose the lemma is false. Then there exists $\varepsilon > 0$ and a subsequence such that, for each α, we may select $g_\alpha \in \mathcal{D}_a^{1,p}(\Omega,0)$ satisfying

$$|\langle J_\infty'(v), \tau_{y_\alpha,\lambda_\alpha} g_\alpha\rangle| > \varepsilon \|g_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)}$$

Relabeling each g_α to be the rescaled $g_\alpha / \|g_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)}$, we obtain

$$|\langle J_\infty'(v), \tau_{y_\alpha,\lambda_\alpha} g_\alpha\rangle| > \varepsilon \tag{2.5.3}$$

for all α where $g_\alpha \in \mathcal{D}_a^{1,p}(\Omega, 0)$ has unit norm.

On the other hand, since $a = b$, it follows from Lemma 2.3.1 that $\tau_{y_\alpha,\lambda_\alpha} g_\alpha$ is a bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. Therefore, a subsequence of $(\tau_{y_\alpha,\lambda_\alpha} g_\alpha)$ converges weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and pointwise almost everywhere to some function $h \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. By Lemma 2.4.3, in fact, $h \in \mathcal{D}^{1,p}(\Omega_\infty)$. Then, by weak convergence and using that v is a critical point of J_∞ in Ω_∞, we have

$$\langle J'_\infty(v), \tau_{y_\alpha,\lambda_\alpha} g_\alpha \rangle \to \langle J'_\infty(v), h \rangle = 0$$

However, this contradicts equation (2.5.3). □

2.5.1 Locality

Due to the nature of the transform $\tau_{y,\lambda}$, restricting domains may have negligible effect on norms (recall Lemma 2.3.5). In relation to the energy functional J_∞, a similar phenomenon can be observed:

Lemma 2.5.3. *Fix $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. If $a = b$ and property (P3) holds (i.e. $|y_\alpha| / \lambda_\alpha \to \infty$) then*

$$R_g(y_\alpha, \lambda_\alpha) := \langle J'_\infty(v), \tau_{y_\alpha,\lambda_\alpha} g \rangle - \left(\int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} \left(\nabla v, \nabla \left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|} \right)^{-ap} \mathrm{d}x \right.$$
$$\left. - \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |v|^{q-2} v \left[\tau_{y_\alpha,\lambda_\alpha} g\right] \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|} \right)^{-bq} \mathrm{d}x \right)$$

tends to 0 uniformly over $g \in \mathcal{D}^{1,p}_a(\Omega, 0)$ as $\alpha \to \infty$.

Proof. For any $g \in \mathcal{D}^{1,p}_a(\Omega, 0)$, observe that

$$\begin{aligned}
R_g(y_\alpha, \lambda_\alpha) = &\int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} \left(\nabla v, \nabla \left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|y_\alpha|^{-ap} - |\lambda_\alpha x + y_\alpha|^{-ap}}{|y_\alpha|^{-ap}} \right) \mathrm{d}x \\
&+ \int_{|x| > \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} \left(\nabla v, \nabla \left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \mathrm{d}x \\
&- \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |v|^{q-2} v \left[\tau_{y_\alpha,\lambda_\alpha} g\right] \left(\frac{|y_\alpha|^{-bq} - |\lambda_\alpha x + y_\alpha|^{-bq}}{|y_\alpha|^{-bq}} \right) \mathrm{d}x \\
&- \int_{|x| > \frac{|y_\alpha|}{4\lambda_\alpha}} |v|^{q-2} v \left[\tau_{y_\alpha,\lambda_\alpha} g\right] \mathrm{d}x \\
=: &\, I_a^{(1)} + I_a^{(2)} - I_b^{(1)} - I_b^{(2)}.
\end{aligned}$$

For all α sufficiently large (so that $|y_\alpha| / \lambda_\alpha > 16$), we may split

$$
\begin{aligned}
I_a^{(1)} &= \int_{|x| \le \sqrt{\frac{|y_\alpha|}{\lambda_\alpha}}} |\nabla v|^{p-2} \left(\nabla v, \nabla \left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|y_\alpha|^{-ap} - |\lambda_\alpha x + y_\alpha|^{-ap}}{|y_\alpha|^{-ap}} \right) \mathrm{d}x \\
&\quad + \int_{\sqrt{\frac{|y_\alpha|}{\lambda_\alpha}} < |x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} \left(\nabla v, \nabla \left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|y_\alpha|^{-ap} - |\lambda_\alpha x + y_\alpha|^{-ap}}{|y_\alpha|^{-ap}} \right) \mathrm{d}x \\
&=: I_a^{(1.1)} + I_a^{(1.2)}
\end{aligned}
$$

It follows from Lemma 2.3.4-(b) with $K = B\left(0, \sqrt{|y_\alpha| / \lambda_\alpha}\right)$ that, for a constant $C > 0$ independent of α, there holds

$$
\begin{aligned}
\left|I_a^{(1.1)}\right| &\le C \sqrt{\frac{|y_\alpha|}{\lambda_\alpha}} \frac{\lambda_\alpha}{|y_\alpha|} \int_{|x| \le \sqrt{\frac{|y_\alpha|}{\lambda_\alpha}}} |\nabla v|^{p-1} \left|\nabla \left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right| \mathrm{d}x \\
&\le C \sqrt{\frac{\lambda_\alpha}{|y_\alpha|}} \left\|\tau_{y_\alpha,\lambda_\alpha} g\right\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \left(\int_{|x| \le \sqrt{\frac{|y_\alpha|}{\lambda_\alpha}}} |\nabla v|^p \, \mathrm{d}x \right)^{(p-1)/p}.
\end{aligned}
$$

by Hölder's inequality. Applying Lemma 2.3.1 with $a = b$ and using that $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is fixed we have, up to a modification of the constant C,

$$
\left|I_a^{(1.1)}\right| \le C \sqrt{\frac{\lambda_\alpha}{|y_\alpha|}} \|g\|_{\mathcal{D}_a^{1,p}(\Omega)} = o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right)
$$

as $\alpha \to \infty$. By the same procedure applied to $I_a^{(1.2)}$ we instead obtain

$$
\begin{aligned}
\left|I_a^{(1.2)}\right| &\le C \|g\|_{\mathcal{D}_a^{1,p}(\Omega)} \left(\int_{\sqrt{\frac{|y_\alpha|}{\lambda_\alpha}} < |x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^p \, \mathrm{d}x \right)^{(p-1)/p} \\
&= o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega)} \right)
\end{aligned}
$$

by the dominated convergence theorem. Combining, we deduce that

$$
I_a^{(1)} = o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega)} \right) \tag{2.5.4}
$$

Similarly, using also the CKN inequality (CKN), we have

$$\left|I_b^{(1)}\right| = o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega)}\right) \tag{2.5.5}$$

Moreover, using Hölder's inequality, Lemma 2.3.1, and the CKN inequality (CKN), we also have

$$I_a^{(2)} + I_b^{(2)} = o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega)}\right) \qquad \text{as } \alpha \to \infty.$$

by the dominated convergence theorem. Combining the last three equations concludes our proof. □

2.6 Iterative results and Solution Extraction

Maintaining the notation of the previous section, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain containing the origin and y_α in $\overline{\Omega}$, λ_α in $(0,\infty)$ are sequences that will be assumed to satisfy some combination of properties (P1)-(P5).

2.6.1 Extracting a Solution

To establish Theorems 1.1.1-1.1.2, the first step is to extract a (possibly trivial) solution to (A) from a given Palais-Smale sequence. We recall that (u_α) in $\mathcal{D}_a^{1,p}(\Omega,0)$ is called a (PS)-sequence for problem (A) provided it has bounded energy (meaning $J(u_\alpha)$ is bounded) and $J'(u_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(\Omega,0)$. The following result borrows ideas from the unweighted proof presented by Mercuri-Willem [52, Lemma 3.5].

Lemma 2.6.1. *Let (u_α) in $\mathcal{D}_a^{1,p}(\Omega,0)$ be a Palais-Smale sequence for the energy functional J such that,*

(i) $u_\alpha \rightharpoonup u$ *in* $\mathcal{D}_a^{1,p}(\Omega,0)$;

(ii) $u_\alpha \to u$ *a.e. on* Ω;

(iii) $J(u_\alpha) \to c$

as $\alpha \to \infty$. Passing to a subsequence if necessary, there holds

(a) $\nabla u_\alpha \to \nabla u$ *a.e. on* Ω *and* $J'(u) = 0$;

(b) $\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|u_\alpha - u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} = \|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} + o(1)$;

(c) $J(u_\alpha - u) \to c - J(u)$ *as* $\alpha \to \infty$.

(d) $J'(u_\alpha - u) \to 0$ *strongly in* $\mathcal{D}^{-1,p'}_a(\Omega, 0)$.

Proof. Let us define T by equation the equation (2.1.3). Using that (u_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(\Omega, 0)$, we see that $(T(u_\alpha - u))$ is also a bounded sequence in $\mathcal{D}^{1,p}_a(\Omega, 0)$. In particular, up to a subsequence, $(T(u_\alpha - u))$ converges weakly in $\mathcal{D}^{1,p}_a(\Omega, 0)$ and pointwise almost everywhere to some function in $\mathcal{D}^{1,p}_a(\Omega, 0)$. Since $u_\alpha \to u$ pointwise a.e., this function must be the trivial function. That is, $T(u_\alpha - u) \rightharpoonup 0$ weakly in $\mathcal{D}^{1,p}_a(\Omega, 0)$ and pointwise almost everywhere.

With this, let us write

$$\begin{aligned}\Gamma_\alpha &:= \int_\Omega \left(\left[|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u \right], \nabla T(u_\alpha - u) \right) |x|^{-ap} \, \mathrm{d}x \\ &= \langle J'(u_\alpha), T(u_\alpha - u) \rangle + \int_\Omega \left(|u_\alpha|^{q-2} u_\alpha, T(u_\alpha - u) \right) |x|^{-bq} \, \mathrm{d}x \\ &\quad - \int_\Omega \left(|\nabla u|^{p-2} \nabla u, \nabla T(u_\alpha - u) \right) |x|^{-ap} \, \mathrm{d}x.\end{aligned}$$

Since $J'(u_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}_a(\Omega, 0)$ and $(T(u_\alpha - u))$ is bounded in $\mathcal{D}^{1,p}_a(\Omega, 0)$, the first term after the last equality tends to 0. Next, observe that, $\|T(u_\alpha - u)\|_{L^q_b(\Omega,0)} \to 0$ by the dominated convergence theorem and, by the CKN-inequality, (u_α) in $L^q_b(\Omega, 0)$ is a bounded sequence. Therefore, an application of Hölder's inequality shows that the second term also tends to 0. Finally, the third term tends to 0 since $(T(u_\alpha - u)) \rightharpoonup 0$ weakly in $\mathcal{D}^{1,p}_a(\Omega, 0)$.

We conclude that $\Gamma_\alpha \to 0$ as $\alpha \to \infty$. Thus, we may apply Corollary 2.1.2 to see that, up to a subsequence,

1. $\nabla u_\alpha \to \nabla u$ almost everywhere on Ω,

2. $\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|u_\alpha - u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} = \|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} + o(1)$,

3. and
$$|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u_\alpha - \nabla u|^{p-2} (\nabla u_\alpha - \nabla u) \to |\nabla u|^{p-2} \nabla u$$

strongly in $L^{p'}(\Omega, |x|^{-ap})$, where $p' = p/(p-1)$,

4. and
$$|u_\alpha|^{q-2} u_\alpha - |u_\alpha - u|^{q-2} (u_\alpha - u) \to |u|^{q-2} u$$
strongly in $L^{q/(q-1)}(\Omega, |x|^{-bq})$.

as $\alpha \to \infty$. In particular, we have established (a) and (b).

Conclusion (c) follows from the Brézis-Lieb lemma. Indeed,

$$\begin{aligned} J(u_\alpha - u) &= \frac{1}{p} \|u_\alpha - u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \frac{1}{q} \|u_\alpha - u\|^q_{L^q(\Omega,|x|^{-bq})} \\ &= J(u_\alpha) - J(u) + o(1) \\ &\to c - J(u). \end{aligned}$$

Next, let $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$ be arbitrary. By Hölder's inequality followed by (3) and (4), we have

$$\langle J'(u_\alpha - u) - J'(u_\alpha) + J'(u), h\rangle = o\left(\|h\|_{\mathcal{D}^{1,p}_a(\Omega,0)}\right)$$

as $\alpha \to \infty$. On the other hand, by standard convergence results in Lebesgue spaces,
$$\langle J'(u), h\rangle = \lim_{\alpha\to\infty} \langle J'(u_\alpha), h\rangle = 0$$
where the second equality holds since $J'(u_\alpha) \to 0$ strongly in $\mathcal{D}^{-1,p'}_a(\Omega, 0)$. Combining the last two equations, we see tha

$$\langle J'(u_\alpha - u), h\rangle = \langle J'(u_\alpha), h\rangle + o\left(\|h\|_{\mathcal{D}^{1,p}_a(\Omega,0)}\right)$$

as $\alpha \to \infty$. Since $J'(u_\alpha) \to 0$ strongly in $\mathcal{D}^{1,p}_a(\Omega, 0)$, we deduce (d). □

2.6.2 Iterative Results

The proof of the main results, namely Theorems 1.1.1-1.1.2, is iterative. Therefore, central to the proof of the main results of this paper are two iteration results which will enable us to proceed to the next iteration stage.

The first handles weighted bubbles and, in the main theorem, will be used provided condition (P2) holds. In this case, we introduce notation for a translated analog of problem (B),

$$\begin{cases} -\operatorname{div}\left(|x+x_0|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x+x_0|^{-bq}|u|^{q-2}u \quad \text{in } \mathbb{R}^n \\ u \in \mathcal{D}_a^{1,p}\left(\mathbb{R}^n, -x_0\right). \end{cases} \tag{2.6.1}$$

In relation, we define the associated energy functional $J_{x_0,\infty} : \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0) \to \mathbb{R}$ given by

$$J_{x_0,\infty}(u) := \int_{\mathbb{R}^n} |\nabla u|^p |z+x_0|^{-ap}\,\mathrm{d}z - \int_{\mathbb{R}^n} |u|^q |z+x_0|^{-bq}\,\mathrm{d}z,$$

We note that a function $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ is a (weak) solution to (2.6.1) provided $J'_{x_0,\infty}(v) = 0$.

The second iteration result treats unweighted bubbling phenomena in the case (P3) holds. We note that the proofs of the iterative results are very similar. However, due to the presence of a non-homogeneous transform, the latter presents more technical challenges.

Proposition 2.6.2. *Let (u_α) in $\mathcal{D}_a^{1,p}(\Omega, 0)$ be a bounded sequence and fix some $x_0 \in \mathbb{R}^n$ along with $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$. For each $\alpha \in \mathbb{N}$, put*

$$v_\alpha(x) := \lambda_\alpha^\gamma u_\alpha\left(\lambda_\alpha(x+x_0)\right)$$

and suppose that

(i) $\lambda_\alpha \to 0$, i.e. (P1) holds,

(ii) $J(u_\alpha) \to c$ and $J'(u_\alpha) \to 0$ strongly in the dual of $\mathcal{D}_a^{1,p}(\Omega, 0)$,

(iii) $v_\alpha \rightharpoonup v$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ and pointwise almost everywhere,

as $\alpha \to \infty$. Then, passing to a subsequence if necessary, $\nabla v_\alpha \to \nabla v$ almost everywhere and $J'_{x_0,\infty}(v) = 0$. Finally, the sequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ given by

$$w_\alpha(z) := u_\alpha(z) - \lambda_\alpha^{-\gamma} v\left(\frac{z}{\lambda_\alpha} - x_0\right)$$

satisfies

(a) $\|w_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} = \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} - \|v\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} + o(1)$,

(b) $J_{0,\infty}(w_\alpha) \to c - J_{x_0,\infty}(v)$,

(c) $J'_{0,\infty}(w_\alpha) \to 0$ *strongly in the dual of* $\mathcal{D}^{1,p}_a(\Omega, 0)$,

as $\alpha \to \infty$.

When $a < b$ in the problem parameters, this last proposition is the only iterative result required to establish the conclusions of Theorem 1.1.1. In the general setting, we shall further require the following:

Proposition 2.6.3. *Suppose we are given a bounded sequence* (u_α) *in* $\mathcal{D}^{1,p}_a(\Omega, 0)$ *along with a set* Ω_∞*, which is either* $\mathbb{R}^n$ *or a half-space, and* $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ *such that the following hold as* $\alpha \to \infty$*:*

(i) $J(u_\alpha) \to c$ *and* $J'(u_\alpha) \to 0$ *strongly in the dual of* $\mathcal{D}^{1,p}_a(\Omega, 0)$,

(ii) $\Omega_\alpha = (\Omega - y_\alpha)/\lambda_\alpha$ *tends to* Ω_∞,

(iii) $v_\alpha \rightharpoonup v$ *weakly in* $\mathcal{D}^{1,p}(\mathbb{R}^n)$ *and pointwise almost everywhere,*

where $v_\alpha := \tau_{y_\alpha,\lambda_\alpha} u_\alpha$*. If* $a = b$ *(so* $q = p^*$*) and (P3) holds (i.e.* $|y_\alpha|/\lambda_\alpha \to \infty$*) then,* $v \in \mathcal{D}^{1,p}(\Omega_\infty)$ *and, up to a subsequence,* $\nabla v_\alpha \to \nabla v$ *almost everywhere. Moreover,* $J'_\infty(v) = 0$ *in* Ω_∞*; i.e.*

$$\langle J'_\infty(v), h\rangle = 0 \qquad \forall h \in \mathcal{D}^{1,p}(\Omega_\infty).$$

Finally, $w_\alpha := u_\alpha - \tau^-_{y_\alpha,\lambda_\alpha} v$ *satisfies*

(a) $\|w_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} = \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|v\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} + o(1)$,

(b) $J_{0,\infty}(w_\alpha) \to c - J_\infty(v)$,

(c) $J'_{0,\infty}(w_\alpha) \to 0$ *strongly in the dual of* $\mathcal{D}^{1,p}_a(\Omega, 0)$.

as $\alpha \to \infty$.

Notice that the inference $J'_\infty(v) = 0$ in Ω_∞ amounts to stating that v solves a PDE in Ω_∞. Specifically, if $\Omega_\infty = \mathbb{R}^n$, then v solves (C), whilst v satisfies (D) if $\Omega_\infty = \mathbb{H}$.

Proof of Proposition 2.6.2. We adapt the argument from Mercuri-Willem [52]. Given $k \in \mathbb{N}$, let B_k denote the open ball $B(0,k) \subset \mathbb{R}^n$. As a first step, we claim that $J'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_k, -x_0)$ as $\alpha \to \infty$. Indeed, fix $h \in C_c^\infty(B_k)$ and define

$$h_\alpha(z) := \lambda_\alpha^{-\gamma} h\left(\frac{z}{\lambda_\alpha} - x_0\right)$$

Observe that, since h_α is supported in $\lambda_\alpha B(x_0,k)$, for all α large $h_\alpha \in C_c^\infty(\Omega)$. Now, by a change of variables,

$$\left\langle J'_{x_0,\infty}(v_\alpha), h\right\rangle = \langle J'(u_\alpha), h_\alpha\rangle$$

for all α large (so that the right-hand side is defined). In particular, using homogeneity, we see that

$$\begin{aligned}\left|\left\langle J'_{x_0,\infty}(v_\alpha), h\right\rangle\right| = |\langle J'(u_\alpha), h_\alpha\rangle| &\le \|J'(u_\alpha)\|_{\mathcal{D}_a^{-1,p'}(\Omega,0)} \|h_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \\ &= \|J'(u_\alpha)\|_{\mathcal{D}_a^{-1,p'}(\Omega,0)} \|h\|_{\mathcal{D}^{1,p}(B_k,-x_0)} .\end{aligned}$$

so $J'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_k, -x_0)$. Using this, we will extract a subsequence such that $\nabla v_\alpha \to \nabla v$ pointwise on $\mathbb{R}^n$, almost everywhere. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a bump function such that

$$\begin{cases} 0 \le \rho \le 1 & \text{on } \mathbb{R}^n, \\ \rho \equiv 1 & \text{in } B_k, \\ \rho \equiv 0 & \text{outside } B_{k+1}. \end{cases}$$

Consider the vector-valued map

$$f_\alpha := |\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v|^{p-2} \nabla v,$$

which satisfies $f_\alpha \cdot (\nabla v_\alpha - \nabla v) \ge 0$ almost everywhere (see Szulkin-Willem [65]).

Let T be as in (2.1.3). For each fixed $k \in \mathbb{N}$, an easy computation shows that

$$\begin{aligned}
0 &\leq \int_{B_k} (f_\alpha, \nabla [T(v_\alpha - v)]) |x + x_0|^{-ap} \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} (f_\alpha, \rho \nabla [T(v_\alpha - v)]) |x + x_0|^{-ap} \, \mathrm{d}x \\
&= \left\langle J'_{x_0,\infty}(v_\alpha), \rho T(v_\alpha - v) \right\rangle \quad + \underbrace{\int_{\mathbb{R}^n} |v_\alpha|^{q-2} v_\alpha \rho T(v_\alpha - v) |x + x_0|^{-bq} \, \mathrm{d}x}_{=:I_\alpha^{(1)}} \\
&\qquad - \underbrace{\int_{\mathbb{R}^n} (f_\alpha, T(v_\alpha - v) \nabla \rho) |x + x_0|^{-ap} \, \mathrm{d}x}_{=:I_\alpha^{(2)}} \\
&\qquad - \underbrace{\int_{\mathbb{R}^n} |\nabla v|^{p-2} (\nabla v, \nabla [\rho T(v_\alpha - v)]) |x + x_0|^{-ap} \, \mathrm{d}x}_{=:I_\alpha^{(3)}} .
\end{aligned}$$

First, since (v_α) is a bounded sequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$, we see that $(T(v_\alpha - v))$ is also a bounded sequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$. Becasue $\rho \in C_c^\infty(B_{k+1})$, it follows that $(\rho T(v_\alpha - v))$ is a bounded sequence in $\mathcal{D}_a^{1,p}(B_{k+1}, -x_0)$. Since $J'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_{k+1}, -x_0)$, we deduce that

$$\left\langle J'_{x_0,\infty}(v_\alpha), \rho T(v_\alpha - v) \right\rangle \to 0$$

as $\alpha \to \infty$.

Next, by the CKN inequality, (v_α) is also a bounded sequence in $L_b^q(\mathbb{R}^n, -x_0)$. Therefore, by Hölder's inequality followed by the dominated convergence theorem, we see that $I_\alpha^{(2)} \to 0$ as $\alpha \to \infty$.

Next, using again that (v_α) is a bounded sequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$, we see that the sequence (f_α) is bounded in $L^{p'}(\mathbb{R}^n, |x + x_0|^{-ap})$. Therefore, using also Hölder's inequality and the fact that $\operatorname{supp}(\rho) \subseteq B_{k+1}$, it is easily seen that

$$\left| I_\alpha^{(2)} \right| \leq C \left(\int_{B_{k+1}} |T(v_\alpha - v)|^p |x + x_0|^{-ap} \, \mathrm{d}x \right)^{1/p}$$

for some constant $C > 0$. By the Dominated Convergence Theorem, we conclude that the right-hand side tends to 0 so $I_\alpha^{(2)} \to 0$ as $\alpha \to \infty$.

Finally, observe that we can decompose

$$\begin{aligned} I_\alpha^{(3)} = &\int_{\mathbb{R}^n} |\nabla v|^{p-2} (\nabla v, \nabla \rho)\, T(v_\alpha - v)\, |x + x_0|^{-ap} \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n} \rho\, |\nabla v|^{p-2} (\nabla v, \nabla [v_\alpha - v])\, |x + x_0|^{-ap} \,\mathrm{d}x \\ &- \int_{\{|v_\alpha - v| > 1\}} \rho\, |\nabla v|^{p-2} (\nabla v, \nabla [v_\alpha - v])\, |x + x_0|^{-ap} \,\mathrm{d}x. \end{aligned}$$

On the right-hand side, the first term converges to by Hölder's inequality and the Dominated Convergence Theorem. The second integral converges to zero since $v_\alpha \rightharpoonup v$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$. Finally, the third integral also converges to zero by Hölder's inequality and the dominated convergence theorem. Indeed, in absolute value, the third integral is bounded above by

$$\|v_\alpha - v\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)} \left(\int_{\{|v_\alpha - v| > 1\}} |\nabla v|^p \,|x + x_0|^{-ap} \,\mathrm{d}x \right)^{(p-1)/p} \to 0.$$

Therefore, we have also found that $I_\alpha^{(3)} \to 0$ as $\alpha \to \infty$.

Combining our work, we conclude that

$$\lim_{\alpha \to \infty} \int_{B_k} \left(\left[|\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v|^{p-2} \nabla v \right], \nabla \left[T(v_\alpha - v) \right] \right) |x + x_0|^{-ap} \,\mathrm{d}x = 0$$

for each $k \in \mathbb{N}$. Citing Proposition 2.1.1, we may assume after passing to a subsequence that $\nabla v_\alpha \to \nabla v$ pointwise almost everywhere on $\mathbb{R}^n$. Therefore, using also that $v_\alpha \to v$ almost everywhere, it follows from standard convergence results that

$$\left\langle J'_{x_0,\infty}(v), h \right\rangle = \lim_{\alpha \to \infty} \left\langle J'_{x_0,\infty}(v_\alpha), h \right\rangle$$

for any $h \in C_c^\infty(\mathbb{R}^n)$. Since h must be supported in B_k for some k and $J'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_k)$, we conclude that

$$\left\langle J'_{x_0,\infty}(v), h \right\rangle = 0$$

Since $h \in C_c^\infty(\mathbb{R}^n)$ was arbitrary, we conclude that $J'_{x_0,\infty}(v) = 0$.

By Corollary 2.1.2, up to a subsequence, we have

(1) $\|v_\alpha - v\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} = \|v_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} - \|v\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} + o(1);$

(2) and
$$|\nabla v_\alpha|^{p-2}\nabla v - |\nabla v_\alpha - \nabla v|^{p-2}(\nabla v_\alpha - \nabla v) \to |\nabla v|^{p-2}\nabla v$$
in $L^{p'}(\mathbb{R}^n, |x+x_0|^{-ap})$;

(3) and
$$|v_\alpha|^{q-2}v_\alpha - |v_\alpha - v|^{q-2}(v_\alpha - v) \to |v|^{q-2}u$$
strongly in $L^{\frac{q}{q-1}}(\mathbb{R}^n, |x+x_0|^{-bq})$

as $\alpha \to \infty$.

By a simple change of variables, it is clear that (1) implies (a) directly. Next, using the same change of variables, we see that

$$J_{0,\infty}(w_\alpha) = J_{x_0,\infty}(v_\alpha - v)$$

Then, the Brézis-Lieb lemma ensures that

$$\begin{aligned} J_{0,\infty}(w_\alpha) = J_{x_0,\infty}(v_\alpha) - J_{x_0,\infty}(v) + o(1) &= J(u_\alpha) - J_{x_0,\infty}(v) + o(1) \\ &= c - J_{x_0,\infty}(v) + o(1) \end{aligned}$$

where the second equality follows from another change of variables. Thus, we have established (b).

Finally, in order to prove claim (c) we fix an arbitrary $g \in \mathcal{D}^{1,p}_a(\Omega, 0)$. After rescaling, we see that

$$\left\langle J'_{0,\infty}(w_\alpha), g\right\rangle = \left\langle J'_{x_0,\infty}(v_\alpha - v), g_\alpha\right\rangle$$

where $g_\alpha(x) = \lambda_\alpha^\gamma g(\lambda_\alpha(x+x_0))$. Then, applying Hölder's inequality together with

(2) and a change of variables,

$$\begin{aligned}\left\langle J'_{0,\infty}(w_\alpha), g\right\rangle &= \left\langle J'_{x0,\infty}(v_\alpha - v), g_\alpha\right\rangle \\ &= \left\langle J'_{x0,\infty}(v_\alpha), g_\alpha\right\rangle - \left\langle J'_{x0,\infty}(v), g_\alpha\right\rangle + o\left(\|g\|_{\mathcal{D}^{1,p}_a(\Omega,0)}\right) \\ &= \langle J'(u_\alpha), g\rangle + o\left(\|g\|_{\mathcal{D}^{1,p}_a(\Omega,0)}\right)\end{aligned}$$

as $\alpha \to \infty$, over$g \in \mathcal{D}^{1,p}_a(\Omega, 0)$. We infer that $J'_{0,\infty}(w_\alpha) \to 0$ strongly in $\mathcal{D}^{-1,p'}_a(\Omega, 0)$ and the proof is complete. □

We now turn towards results involved in the proof of Proposition 2.6.3, beginning with the following lemma.

Lemma 2.6.4. *Suppose (u_α) in $\mathcal{D}^{1,p}_a(\Omega, 0)$ is a given bounded sequence and there is $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that*

(i) $\tau_{y_\alpha,\lambda_\alpha} u_\alpha \rightharpoonup v$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and pointwise almost everywhere.

(ii) $\nabla\left[\tau_{y_\alpha,\lambda_\alpha} u_\alpha\right] \to \nabla v$ almost everywhere in $\mathbb{R}^n$.

If $a = b$ (so $q = p^*$) and (P3) holds (i.e. $|y_\alpha|/\lambda_\alpha \to \infty$) then $w_\alpha := u_\alpha - \tau^-_{y_\alpha,\lambda_\alpha} v$ satisfies

(a) $\|w_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} = \|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|v\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} + o(1)$ and

(b) $\|w_\alpha\|^q_{L^q_b(\mathbb{R}^n,0)} = \|u_\alpha\|^q_{L^q_b(\Omega,0)} - \|v\|^q_{L^q(\mathbb{R}^n)} + o(1)$

as $\alpha \to \infty$

Proof. For convenience, let us write $\tau^-_\alpha := \tau^-_{y_\alpha,\lambda_\alpha}$ and extend each (u_α) by 0 to all of $\mathbb{R}^n$. In order to establish (a), we split

$$\|w_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n)} = \int_{B_\alpha} |\nabla w_\alpha(z)|^p\, |z|^{-ap}\, \mathrm{d}z + \int_{\mathbb{R}^n\setminus B_\alpha} |\nabla w_\alpha(z)|^p\, |z|^{-ap}\, \mathrm{d}z \qquad (2.6.2)$$

where $B_\alpha := B(y_\alpha, |y_\alpha|/4)$. The idea is that in the ball B_α, the η term is neutralized and, due to Lemma 2.3.5, we expect $\tau^-_\alpha v$ to be negligible outside of B_α.

In order to handle the first term on the right-hand side of equation (2.6.2), we add and subtract terms to obtain

$$\begin{aligned}\int_{B_\alpha} |\nabla w_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z &= \int_{B_\alpha} \left(|\nabla w_\alpha(z)|^p + \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p - |\nabla u_\alpha(z)|^p\right) |z|^{-ap} \, \mathrm{d}z \\ &\quad + \int_{B_\alpha} |\nabla u_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z - \int_{B_\alpha} \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p |z|^{-ap} \, \mathrm{d}z.\end{aligned} \tag{2.6.3}$$

Now, observe that observe that since $a = b$, properties of η and a change of variables yields

$$\begin{aligned}\int_{B_\alpha} \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p |z|^{-ap} \, \mathrm{d}z &= \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v(x)|^p \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|}\right)^{-ap} \mathrm{d}x \\ &= \int_{\mathbb{R}^n} |\nabla v(x)|^p \, \mathrm{d}x + o(1)\end{aligned}$$

as $\alpha \to \infty$ by the dominated convergence theorem (use Lemma 2.3.4-(a) and that $|y_\alpha| / \lambda_\alpha \to \infty$). Returning to equation (2.6.3), we see that, as $\alpha \to \infty$,

$$\begin{aligned}\int_{B_\alpha} |\nabla w_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z &= \int_{B_\alpha} \left(|\nabla w_\alpha(z)|^p + \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p - |\nabla u_\alpha(z)|^p\right) |z|^{-ap} \, \mathrm{d}z \\ &\quad + \int_{B_\alpha} |\nabla u_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z - \int_{\mathbb{R}^n} |\nabla v(x)|^p \, \mathrm{d}x + o(1).\end{aligned} \tag{2.6.4}$$

Moreover, it is easy to show that there exists a constant $C > 0$ such that

$$\left||x|^p + |y|^p - |x+y|^p\right| \le C \left(|x|^{p-1} \, |y| + |x| \, |y|^{p-1}\right) \qquad \forall x, y \in \mathbb{R}^n. \tag{2.6.5}$$

Therefore, we may bound

$$\begin{aligned}&\int_{B_\alpha} \left||\nabla w_\alpha(z)|^p + \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p - |\nabla u_\alpha(z)|^p\right| |z|^{-ap} \, \mathrm{d}z \\ &\le C \int_{B_\alpha} \left(\left|\nabla \left[\tau_\alpha^- v\right](z)\right|^{p-1} |\nabla w_\alpha(z)| + \left|\nabla \left[\tau_\alpha^- v\right](z)\right| |\nabla w_\alpha(z)|^{p-1}\right) |z|^{-ap} \, \mathrm{d}z \\ &\le C \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} \left(|\nabla v|^{p-1} \, |\nabla[v_\alpha - v]| + |\nabla v| \, |\nabla[v_\alpha - v]|^{p-1}\right) \mathrm{d}x\end{aligned}$$

where, with a possible modification of the constant C, the last inequality follows from a change of variables. Leveraging pointwise convergence and boundedness in appropriate Lebesgue spaces, it follows from standard results that this last term is $o(1)$ as $\alpha \to \infty$. Therefore, returning to (2.6.4), we have

$$\int_{B_\alpha} |\nabla w_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z = \int_{B_\alpha} |\nabla u_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z - \int_{\mathbb{R}^n} |\nabla v(x)|^p \, \mathrm{d}x + o(1). \tag{2.6.6}$$

Next, to handle the second term on the right-hand side of equation (2.6.2), we once again add and subtract terms to obtain, as $\alpha \to \infty$,

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_\alpha} |\nabla w_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z \\ &= \int_{\mathbb{R}^n \setminus B_\alpha} \left(|\nabla w_\alpha(z)|^p - |\nabla u_\alpha(z)|^p - \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p \right) |z|^{-ap} \, \mathrm{d}z \\ &\quad + \int_{\mathbb{R}^n \setminus B_\alpha} |\nabla u_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z + \int_{\mathbb{R}^n \setminus B_\alpha} \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p |z|^{-ap} \, \mathrm{d}z. \end{aligned} \tag{2.6.7}$$

By Lemma 2.3.5, the last term on the right-hand side tends to 0 as $\alpha \to \infty$. Furthermore, using once again inequality (2.6.5), we bound

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_\alpha} \left| |\nabla w_\alpha(z)|^p - |\nabla u_\alpha(z)|^p - \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^p \right| |z|^{-ap} \, \mathrm{d}z \\ &\leq C \int_{\mathbb{R}^n \setminus B_\alpha} |\nabla u_\alpha(z)|^{p-1} \left|\nabla \left[\tau_\alpha^- v\right](z)\right| |z|^{-ap} \, \mathrm{d}z \\ &\qquad + C \int_{\mathbb{R}^n \setminus B_\alpha} |\nabla u_\alpha(z)| \left|\nabla \left[\tau_\alpha^- v\right](z)\right|^{p-1} |z|^{-ap} \, \mathrm{d}z \\ &\leq C \left(\|u_\alpha\|^{p-1}_{\mathcal{D}^{1,p}_a(\Omega,0)} \left\|\nabla \left[\tau_\alpha^- v\right]\right\|_{L^p_a(\mathbb{R}^n \setminus B_\alpha)} + \|u_\alpha\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \left\|\nabla \left[\tau_\alpha^- v\right]\right\|^{p-1}_{L^p_a(\mathbb{R}^n \setminus B_\alpha)} \right) \end{aligned}$$

by Hölder's inequality. Since (u_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(\Omega, 0)$, it follows from Lemma 2.3.5 that the last equation tends to 0 as $\alpha \to \infty$. Thus, returning to (2.6.7), we see that

$$\int_{\mathbb{R}^n \setminus B_\alpha} |\nabla w_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z = \int_{\mathbb{R}^n \setminus B_\alpha} |\nabla u_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z + o(1) \tag{2.6.8}$$

Combining equations (2.6.6) and (2.6.8), we obtain

$$\int_{\mathbb{R}^n} |\nabla w_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z = \int_{\mathbb{R}^n} |\nabla u_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}z - \int_{\mathbb{R}^n} |\nabla v(x)|^p \, \mathrm{d}x + o(1)$$

which is precisely (a). We may deduce (b) via a similar argument. □

Lemma 2.6.5. *Suppose (u_α) in $\mathcal{D}^{1,p}_a(\Omega, 0)$ is a given bounded sequence and there is $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that*

(i) $v_\alpha \rightharpoonup v$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and pointwise almost everywhere.

(ii) $\nabla v_\alpha \to \nabla v$ almost everywhere in $\mathbb{R}^n$,

(iii) with $p' = p/(p-1)$ denoting the Hölder conjugate of p,

$$|\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v_\alpha - \nabla v|^{p-2} (\nabla v_\alpha - \nabla v) \to |\nabla v|^{p-2} \nabla v$$

in $L^{p'}(\mathbb{R}^n)$,

(iv) with $q' = q/(q-1)$ denoting the Hölder conjugate of q,

$$|v_\alpha|^{q-2} v_\alpha - |v_\alpha - v|^{q-2} (v_\alpha - v) \to |v|^{q-2} v$$

in $L^{q'}(\mathbb{R}^n)$,

where $v_\alpha := \tau_{y_\alpha, \lambda_\alpha} u_\alpha$. If $a = b$ (so $q = p^*$) and (P3) holds (i.e. $|y_\alpha| / \lambda_\alpha \to \infty$) then, with $w_\alpha := u_\alpha - \tau^{-}_{y_\alpha, \lambda_\alpha} v$,

$$\left\langle J'_{0,\infty}(w_\alpha), g \right\rangle = \langle J'(u_\alpha), g \rangle - \langle J'_\infty(v), \tau_{y_\alpha, \lambda_\alpha} g \rangle + o\left(\|g\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \right)$$

as $\alpha \to \infty$ over $g \in \mathcal{D}^{1,p}_a(\Omega, 0)$.

Proof. Let $g \in \mathcal{D}^{1,p}_a(\Omega, 0)$ and define $B_\alpha := B(y_\alpha, |y_\alpha| / 4)$ as in Lemma 2.6.4. We split as follows:

$$
\begin{aligned}
\left\langle J_{0,\infty}'(w_\alpha), g\right\rangle = &\int_{B_\alpha} |\nabla w_\alpha|^{p-2} (\nabla w_\alpha, \nabla g)\, |z|^{-ap}\,\mathrm{d}z \\
&+ \int_{\mathbb{R}^n \setminus B_\alpha} |\nabla w_\alpha|^{p-2} (\nabla w_\alpha, \nabla g)\, |z|^{-ap}\,\mathrm{d}z \\
&- \int_{B_\alpha} |w_\alpha|^{q-2}\, w_\alpha g\, |z|^{-bq}\,\mathrm{d}z \\
&- \int_{\mathbb{R}^n \setminus B_\alpha} |w_\alpha|^{q-2}\, w_\alpha g\, |z|^{-bq}\,\mathrm{d}z \\
=: &\; I_a^{(1)} + I_a^{(2)} - I_b^{(1)} - I_b^{(2)}.
\end{aligned}
$$

On B_α, using properties of η and performing a change of variables, we see that

$$
I_a^{(1)} = \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla[v_\alpha - v]|^{p-2} \left(\nabla[v_\alpha - v], \nabla\left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|}\right)^{-ap} \mathrm{d}x
$$

Using this expression for $I_a^{(1)}$, it follows from Lemma 2.3.4-(a) that there exists a constant $C > 0$ such that

$$
\begin{aligned}
&\left| I_a^{(1)} - \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} \left(\left[|\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v|^{p-2} \nabla v\right], \nabla\left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|}\right)^{-ap} \mathrm{d}x \right| \\
&\le C \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} \left| |\nabla[v_\alpha - v]|^{p-2} \nabla[v_\alpha - v] - |\nabla v_\alpha|^{p-2} \nabla v_\alpha + |\nabla v|^{p-2} \nabla v \right| \cdot \left|\nabla\left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right| \mathrm{d}x \\
&= o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)}\right)
\end{aligned}
$$

as $\alpha \to \infty$. Here, the last line is derived using Hölder's inequality followed by property (iii) and Lemma 2.3.1. From this, we have

$$
\begin{aligned}
I_a^{(1)} = &\int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v_\alpha|^{p-2} \left(\nabla v_\alpha, \nabla\left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|}\right)^{-ap} \mathrm{d}x \\
&- \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} \left(\nabla v, \nabla\left[\tau_{y_\alpha,\lambda_\alpha} g\right]\right) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|}\right)^{-ap} \mathrm{d}x + o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)}\right)
\end{aligned}
$$

as $\alpha \to \infty$. After a change of variables on the first integral, we obtain

$$I_a^{(1)} = \int_{B_\alpha} |\nabla u_\alpha|^{p-2} (\nabla u_\alpha, \nabla g) |z|^{-ap} \, \mathrm{d}z \qquad (2.6.9)$$
$$- \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} (\nabla v, \nabla [\tau_{y_\alpha,\lambda_\alpha} g]) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|} \right)^{-ap} \mathrm{d}x + o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right)$$

as $\alpha \to \infty$. Similarly, using property (iv) instead of property (iii), we have

$$I_b^{(1)} = \int_{B_\alpha} |u_\alpha|^{q-2} u_\alpha(z) g(z) |z|^{-bq} \, \mathrm{d}z \qquad (2.6.10)$$
$$- \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |v|^{q-2} v(x) [\tau_{y_\alpha,\lambda_\alpha} g](x) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|} \right)^{-bq} \mathrm{d}x + o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right)$$

as $\alpha \to \infty$.

Next, using Hölder's inequality, we may bound

$$\begin{aligned} &\left| I_a^{(2)} - \int_{\mathbb{R}^n \setminus B_\alpha} |\nabla u_\alpha|^{p-2} (\nabla u_\alpha, \nabla g) |z|^{-ap} \, \mathrm{d}z \right| \\ &= \left| \int_{\mathbb{R}^n \setminus B_\alpha} \left(\left[|\nabla w_\alpha|^{p-2} \nabla w_\alpha - |\nabla u_\alpha|^{p-2} \nabla u_\alpha \right], \nabla g \right) |z|^{-ap} \, \mathrm{d}z \right| \qquad (2.6.11) \\ &\le \left\| |\nabla w_\alpha|^{p-2} \nabla w_\alpha - |\nabla u_\alpha|^{p-2} \nabla u_\alpha \right\|_{L_a^{p'}(\mathbb{R}^n \setminus B_\alpha)} \|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \end{aligned}$$

Let us now bound the right hand side by considering two distinct cases. First, if $1 < p \le 2$ then it follows from a result of Mercuri-Willem [52, Lemma 3.1] that, for some constant $C > 0$,

$$\left| |\nabla w_\alpha|^{p-2} \nabla w_\alpha - |\nabla u_\alpha|^{p-2} \nabla u_\alpha \right| \le C |\nabla w_\alpha - \nabla u_\alpha|^{p-1} = C \left| \nabla \left[\tau_{y_\alpha,\lambda_\alpha}^{-} v \right] \right|^{p-1}.$$

Then,

$$\left\| |\nabla w_\alpha|^{p-2} \nabla w_\alpha - |\nabla u_\alpha|^{p-2} \nabla u_\alpha \right\|_{L_a^{p'}(\mathbb{R}^n \setminus B_\alpha)} \le C \left\| \nabla \left[\tau_{y_\alpha,\lambda_\alpha}^{-} v \right] \right\|_{L_a^{p}(\mathbb{R}^n \setminus B_\alpha)} \to 0$$

as $\alpha \to \infty$ by Lemma 2.3.5. In conclusion, when $1 < p \le 2$, we have shown that

$$\left\| |\nabla w_\alpha|^{p-2} \nabla w_\alpha - |\nabla u_\alpha|^{p-2} \nabla u_\alpha \right\|_{L_a^{p'}(\mathbb{R}^n \setminus B_\alpha)} \to 0.$$

On the other hand, if $p > 2$ then it is easy to see that there is a constant $C = C(p) > 0$ such that

$$\left||x+y|^{p-2}(x+y) - |x|^{p-2}x - |y|^{p-2}y\right| \leq C\left(|x|^{p-2}|y| + |x||y|^{p-2}\right), \quad \forall x, y \in \mathbb{R}^n.$$

Therefore,

$$\begin{aligned}
&\left||\nabla w_\alpha|^{p-2}\nabla w_\alpha - |\nabla u_\alpha|^{p-2}\nabla u_\alpha - \left|-\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right|^{p-2}\left(-\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right)\right| \\
&\leq C\left(|u_\alpha|^{p-2}\left|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right| + |u_\alpha|\left|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right|^{p-2}\right)
\end{aligned}$$

Then, the triangle inequality followed by this last identity allows us bound as follows:

$$\begin{aligned}
&\left\||\nabla w_\alpha|^{p-2}\nabla w_\alpha - |\nabla u_\alpha|^{p-2}\nabla u_\alpha\right\|_{L^{p'}_a(\mathbb{R}^n\setminus B_\alpha)} \\
&\leq \left\||\nabla w_\alpha|^{p-2}\nabla w_\alpha - |\nabla u_\alpha|^{p-2}\nabla u_\alpha + \left|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right|^{p-2}\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right\|_{L^{p'}_a(\mathbb{R}^n\setminus B_\alpha)} \\
&\quad + \left\|\left|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right|^{p-1}\right\|_{L^{p'}_a(\mathbb{R}^n\setminus B_\alpha)} \\
&\leq C\left(\left\||\nabla u_\alpha|^{p-2}\left|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right|\right\|_{L^{p'}_a(\mathbb{R}^n\setminus B_\alpha)} + \left\||\nabla u_\alpha|\left|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right|^{p-2}\right\|_{L^{p'}_a(\mathbb{R}^n\setminus B_\alpha)}\right) \\
&\quad + \left\|\nabla\left[\tau^-_{y_\alpha,\lambda_\alpha}v\right]\right\|_{L^p_a(\mathbb{R}^n\setminus B_\alpha)}
\end{aligned}$$

Then, using Hölder's inequality (with exponent $p-1 > 1$ and conjugate exponent $(p-1)/(p-2)$) and that (u_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(\Omega, 0)$, it follows from Lemma 2.3.5 that

$$\left\||\nabla w_\alpha|^{p-2}\nabla w_\alpha - |\nabla u_\alpha|^{p-2}\nabla u_\alpha\right\|_{L^{p'}_a(\mathbb{R}^n\setminus B_\alpha)} \to 0.$$

Having demonstrated this convergence both in the case $1 < p \leq 2$ and $p > 2$, we return to equation (2.6.11) and conclude that

$$I^{(2)}_a = \int_{\mathbb{R}^n\setminus B_\alpha} |\nabla u_\alpha|^{p-2}\nabla(u_\alpha, \nabla g)\,|z|^{-ap}\,\mathrm{d}z + o\left(\|g\|_{\mathcal{D}^{1,p}_a(\Omega,0)}\right)$$

as $\alpha \to \infty$. Similarly, we obtain

$$I_b^{(2)} = \int_{\mathbb{R}^n \setminus B_\alpha} |u_\alpha|^{q-2} u_\alpha g \, |z|^{-bq} \, \mathrm{d}z + o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right)$$

as $\alpha \to \infty$.

Combining these last two equations with (2.6.9) and (2.6.10), as $\alpha \to \infty$

$$\begin{aligned}
\left\langle J'_{0,\infty}(w_\alpha), g \right\rangle &= I_a^{(1)} + I_a^{(2)} - I_b^{(1)} - I_b^{(2)} \\
&= \int_{\mathbb{R}^n} |\nabla u_\alpha|^{p-2} (\nabla u_\alpha, \nabla g) \, |z|^{-ap} \, \mathrm{d}z - \int_{\mathbb{R}^n} |u_\alpha|^{q-2} u_\alpha(z) g(z) \, |z|^{-bq} \, \mathrm{d}z \\
&\quad - \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |\nabla v|^{p-2} (\nabla v, \nabla [\tau_{y_\alpha,\lambda_\alpha} g]) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|} \right)^{-ap} \mathrm{d}x \\
&\quad + \int_{|x| \le \frac{|y_\alpha|}{4\lambda_\alpha}} |v|^{q-2} v(x) \, [\tau_{y_\alpha,\lambda_\alpha} g](x) \left(\frac{|\lambda_\alpha x + y_\alpha|}{|y_\alpha|} \right)^{-bq} \mathrm{d}x \\
&\quad + o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right).
\end{aligned}$$

Finally, by Lemma 2.5.3,

$$\left\langle J'_{0,\infty}(w_\alpha), g \right\rangle = \langle J'(u_\alpha), g \rangle - \langle J'_\infty(v), \tau_{y_\alpha,\lambda_\alpha} g \rangle + o\left(\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \right).$$

□

Proof of Proposition 2.6.3. It immediately follows from Lemma 2.4.3 that $v \in \mathcal{D}^{1,p}(\Omega_\infty)$. Since Ω_∞ is either a half-space or $\mathbb{R}^n$, we may find an increasing collection $(B_k)_{k\in\mathbb{N}}$ of open sets such that

1. B_k is compactly contained in Ω_∞ for each k;

2. For each k, the closure of B_k is a compact set contained in B_{k+1};

3. Every $x \in \Omega_\infty$ is in B_k for all k large.

For fixed k, it follows from Lemma 2.5.1 that

$$\langle J'_\infty(v_\alpha), h \rangle = \left\langle J'(u_\alpha), \tau^{-}_{y_\alpha,\lambda_\alpha} h \right\rangle + o\left(\|h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \right)$$

as $\alpha \to \infty$ over $h \in \mathcal{D}^{1,p}(B_k)$. Here, we note that there is a slight abuse in notation since the right-hand side is only well-defined for α large. Indeed, for all α large, $B_k \subseteq \Omega_\alpha$ so $\tau^-_{y_\alpha,\lambda_\alpha} h \in \mathcal{D}^{1,p}_a(\Omega, 0)$ for each $h \in \mathcal{D}^{1,p}(B_k)$. We may further bound

$$\begin{aligned}
|\langle J'_\infty(v_\alpha), h\rangle| &\leq \left|\left\langle J'(u_\alpha), \tau^-_{y_\alpha,\lambda_\alpha} h\right\rangle\right| + o\left(\|h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)}\right) \\
&\leq \|J'(u_\alpha)\|_{\mathcal{D}^{-1,p'}_a(\Omega,0)} \left\|\tau^-_{y_\alpha,\lambda_\alpha} h\right\|_{\mathcal{D}^{1,p}_a(\Omega,0)} + o\left(\|h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)}\right) \\
&= o\left(\|h\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)}\right)
\end{aligned}$$

as $\alpha \to \infty$ over $h \in \mathcal{D}^{1,p}(B_k)$. Note that, for the last equality, we used the assumption $J'(u_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}_a(\Omega, 0)$ and Lemma 2.3.2. From our work thus far, we see that

$$J'_\infty(v_\alpha) \to 0 \text{ strongly in } \mathcal{D}^{-1,p'}(B_k) \tag{2.6.12}$$

for each fixed $k \in \mathbb{N}$.

Next, we will prove that the conditions of Corollary 2.1.2 are satisfied for the sequence (v_α) with the increasing sequence of open subsets $B_k \nearrow \Omega_\infty$. First, it is clear from Lemma 2.3.1 that (v_α) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. Let $T : \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function

$$T(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ \frac{x}{|x|} & \text{if } |x| > 1. \end{cases}$$

Fix $k \in \mathbb{N}$ and let $\rho \in C^\infty_c(\Omega_\infty) \subseteq C^\infty_c(\mathbb{R}^n)$ be a bump function such that

$$\begin{cases} 0 \leq \rho \leq 1 & \text{in } \mathbb{R}^n \\ \rho \equiv 1 & \text{in } B_k \\ \rho \equiv 0 & \text{outside } B_{k+1}. \end{cases}$$

Observe that

$$f_\alpha := |\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v|^{p-2} \nabla v$$

is uniformly (in α) bounded in $L^{p'}(\mathbb{R}^n)$, with $p' := p/(p-1)$. Moreover, by a classical result (see, for instance, Szulkin-Willem [65, Lemma 2.1] and Simon [61, p.210-212] for the proof), we have that $(f_\alpha, [\nabla v_\alpha - \nabla v]) \geq 0$ almost everywhere. Therefore,

$$\begin{aligned} 0 \leq \int_{B_k} (f_\alpha, \nabla T(v_\alpha - v))\,\mathrm{d}x &\leq \int_{\mathbb{R}^n} (f_\alpha, \rho \nabla T(v_\alpha - v))\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} (f_\alpha, \nabla\left[\rho T(v_\alpha - v)\right])\,\mathrm{d}x - \int_{\mathbb{R}^n} (f_\alpha, T(v_\alpha - v)\nabla\rho)\,\mathrm{d}x \\ &=: I_\alpha^{(1)} - I_\alpha^{(2)}. \end{aligned}$$

We claim that both $I_\alpha^{(1)}$ and $I_\alpha^{(2)}$ tend to 0 as $\alpha \to \infty$. First, by Hölder's inequality,

$$\left|I_\alpha^{(2)}\right| \leq \|f_\alpha\|_{L^{p'}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\nabla\rho|^p\, |T(v_\alpha - v)|^p\,\mathrm{d}x\right)^{1/p}$$

Since the right integrand is bounded by a constant multiple of the indicator function of B_{k+1}, the dominated convergence theorem (using that $v_\alpha \to v$ almost everywhere) implies that $I_\alpha^{(2)} \to 0$ as $\alpha \to \infty$.

Next, in order to estimate $I_\alpha^{(1)}$, we first observe that $\rho T(v_\alpha - v) \in \mathcal{D}^{1,p}(B_{k+1})$ and, in fact, $\rho T(v_\alpha - v)$ is bounded in $\mathcal{D}^{1,p}(B_{k+1})$ uniformly in α. Since $J'_\infty(v_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}(B_{k+1})$,

$$\langle J'_\infty(v_\alpha), \rho T(v_\alpha - v)\rangle \to 0 \quad \text{as } \alpha \to \infty.$$

Therefore, writing

$$\begin{aligned} I_\alpha^{(1)} = \langle J'_\infty(v_\alpha), \rho T(v_\alpha - v)\rangle &+ \underbrace{\int_{\mathbb{R}^n} |v_\alpha|^{q-2}\, v_\alpha \rho T(v_\alpha - v)\mathrm{d}x}_{=:J_\alpha^{(1)}} \\ &- \underbrace{\int_{\mathbb{R}^n} |\nabla v|^{p-2}\left(\nabla v, \nabla\left[\rho T(v_\alpha - v)\right]\right)\mathrm{d}x}_{=:J_\alpha^{(2)}}, \end{aligned}$$

we have

$$I_\alpha^{(1)} = o(1) + J_\alpha^{(1)} - J_\alpha^{(2)}$$

as $\alpha \to \infty$. Using Hölder's inequality, we see that

$$\left|J_\alpha^{(1)}\right| \leq \|v_\alpha\|_{L^q(\mathbb{R}^n)}^{q-1} \left(\int_{\mathbb{R}^n} \rho^q(x) \left|T(v_\alpha - v)\right|^q \mathrm{d}x \right)^{1/q} = o(1)$$

as $\alpha \to \infty$. Here, we have used the dominated convergence theorem and applied the CKN inequality (CKN) to deduce that $\|v_\alpha\|_{L^q(\mathbb{R}^n)}$ is bounded. On the other hand, as $\alpha \to \infty$,

$$\begin{aligned}
J_\alpha^{(2)} &= \int_{\mathbb{R}^n} |\nabla v|^{p-2} \left(\nabla v, \nabla\left[\rho T(v_\alpha - v)\right]\right) \mathrm{d}x \\
&= \int_{|v_\alpha - v| \leq 1} |\nabla v|^{p-2} \left(\nabla v, \rho \nabla (v_\alpha - v)\right) \mathrm{d}x + \int_{\mathbb{R}^n} |\nabla v|^{p-2} \left(\nabla v, T(v_\alpha - v) \nabla \rho\right) \mathrm{d}x \\
&= \int_{|v_\alpha - v| \leq 1} |\nabla v|^{p-2} \left(\nabla v, \rho \nabla (v_\alpha - v)\right) \mathrm{d}x + o(1)
\end{aligned}$$

where, to deduce the last equality, we have used Hölder's inequality and the dominated convergence theorem. We then split the right hand-side as follows:

$$J_\alpha^{(2)} = \int_{\mathbb{R}^n} |\nabla v|^{p-2} \left(\nabla v, \rho \nabla (v_\alpha - v)\right) \mathrm{d}x - \int_{|v_\alpha - v| > 1} |\nabla v|^{p-2} \left(\nabla v, \rho \nabla (v_\alpha - v)\right) \mathrm{d}x + o(1).$$

Note that the first term on the right-hand side tends to 0 as $\alpha \to \infty$. Indeed, this is because $v_\alpha \rightharpoonup v$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ or, equivalently, $\nabla v_\alpha \rightharpoonup \nabla v$ weakly in $L^p(\mathbb{R}^n)$.

$$J_\alpha^{(2)} = -\int_{|v_\alpha - v| > 1} |\nabla v|^{p-2} \left(\nabla v, \rho \nabla (v_\alpha - v)\right) \mathrm{d}x + o(1)$$

With another application of Hölder's inequality we see that,

$$\begin{aligned}
\left|J_\alpha^{(2)}\right| &\leq \int_{\{|v_\alpha - v| > 1\} \cap B_{k+1}} |\nabla v|^{p-1} \left|\nabla (v_\alpha - v)\right| \mathrm{d}x + o(1) \\
&\leq \|v_\alpha - v\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \left(\int_{\{|v_\alpha - v| > 1\} \cap B_{k+1}} |\nabla v|^p \, \mathrm{d}x \right)^{1/p'} + o(1)
\end{aligned}$$

as $\alpha \to \infty$. Since $v_\alpha \to v$ pointwise a.e., the dominated convergence theorem ensures that the right hand side tends to 0 as $\alpha \to \infty$. Especially, $J_\alpha^{(2)} \to 0$ as

$\alpha \to \infty$.

Combining our result for $J_\alpha^{(1)}$ and $J_\alpha^{(2)}$, we see that $I_\alpha^{(1)} \to 0$ as $\alpha \to \infty$. Putting this together with $I_\alpha^{(2)} \to 0$ as $\alpha \to \infty$, we have shown that

$$\int_{B_k \cap \Omega_\infty} (f_\alpha, \nabla T(v_\alpha - v))\, \mathrm{d}x \to 0 \quad \text{as } \alpha \to \infty$$

where

$$f_\alpha = |\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v|^{p-2} \nabla v.$$

Therefore, by Corollary 2.1.2 (invoked with $a = b = 0$), up to a subsequence, we have the following.

(1) $\nabla v_\alpha \to \nabla v$ almost everywhere in Ω_∞. Using that v_α is supported in Ω_α, $v_\alpha \to v$ pointwise almost everywhere and $\Omega_\alpha \to \Omega_\infty$, it readily follows that $\nabla v_\alpha \to \nabla v$ almost everywhere in $\mathbb{R}^n$.

(2) $\|v\|^p_{\mathcal{D}^{1,p}(\Omega_\infty)} = \|v_\alpha\|^p_{\mathcal{D}^{1,p}(\Omega_\infty)} - \|v_\alpha - v\|^p_{\mathcal{D}^{1,p}(\Omega_\infty)} + o(1)$.

(3) There holds

$$|\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v_\alpha - \nabla v|^{p-2} (\nabla v_\alpha - \nabla v) \to |\nabla v|^{p-2} \nabla v.$$

strongly in $L^{p'}(\Omega_\infty)$. Then, since $v \in \mathcal{D}^{1,p}(\Omega_\infty)$ it immediately follows that the convergence remains true in $L^{p'}(\mathbb{R}^n)$.

(4) Finally,

$$|v_\alpha|^{q-2} v_\alpha - |v_\alpha - v|^{q-2} (v_\alpha - v) \to |v|^{q-2} v.$$

strongly in $L^{q/(q-1)}(\Omega_\infty)$ and hence in $L^{q/(q-1)}(\mathbb{R}^n)$.

Next, fix $h \in C_c^\infty(\Omega_\infty)$. Then, $h \in C_c^\infty(B_k)$ for all k large; fix such an index k. Since, as stated in equation (2.6.12), $J'(v_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}(B_k)$, there holds

$$\begin{aligned}
0 &= \lim_{\alpha\to\infty} \langle J'_\infty(v_\alpha), h\rangle \\
&= \lim_{\alpha\to\infty} \left(\int_{\mathbb{R}^n} |\nabla v_\alpha|^{p-2} (\nabla v_\alpha, \nabla h)\, \mathrm{d}x - \int_{\mathbb{R}^n} |v_\alpha|^{q-2} v_\alpha h \mathrm{d}x \right) \\
&= \int_{\mathbb{R}^n} |\nabla v|^{p-2} (\nabla v, \nabla h)\, \mathrm{d}x - \int_{\mathbb{R}^n} |v|^{q-2} v h \mathrm{d}x \\
&= \langle J'_\infty(v), h\rangle
\end{aligned}$$

where the second equality is due to a standard convergence result, using that $v_\alpha \to v$ and $\nabla v_\alpha \to \nabla v$ pointwise almost everywhere. Since $h \in C_c^\infty(\Omega_\infty)$ was arbitrary, we conclude that $J'_\infty(v) = 0$ in Ω_∞.

Next, we establish the desired properties of the sequence (w_α). First, (a) follows from Lemma 2.6.4. By this same lemma, we also see that

$$\begin{aligned}
J_{0,\infty}(w_\alpha) = \frac{1}{p} \|w_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} - \frac{1}{q} \|w_\alpha\|_{L^q_b(\mathbb{R}^n,0)} &= J(u_\alpha) - J_\infty(v) + o(1) \\
&= c - J_\infty(v) + o(1)
\end{aligned}$$

as $\alpha \to \infty$. That is, we have (b). Finally, by Lemma 2.6.5,

$$\left\langle J'_{0,\infty}(w_\alpha), g \right\rangle = \langle J'(u_\alpha), g\rangle - \langle J'_\infty(v), \tau_{y_\alpha,\lambda_\alpha} g\rangle + o\left(\|g\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \right)$$

as $\alpha \to \infty$ over $g \in \mathcal{D}^{1,p}_a(\Omega)$. Then, using that $J'(u_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}_a(\Omega)$ and applying Lemma 2.5.2, we conclude that

$$\left\langle J'_{0,\infty}(w_\alpha), g \right\rangle = o\left(\|g\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \right)$$

which gives (c).

□

2.6.3 Bubbling

In the iteration lemmas, the terms subtracted from the Palais-Smale sequence (u_α) are referred to as "bubbles". We claim that, as $\alpha \to \infty$, the bubble terms converge to 0 in suitable senses. Explicitly, we provide two results.

First, in the first iteration result, i.e. Proposition 2.6.2, the subtracted terms are

$$B_\alpha(z) := \lambda_\alpha^{-\gamma} v\left(\frac{z}{\lambda_\alpha} - x_0\right). \tag{2.6.13}$$

Now, with condition (P1), we have the following result:

Lemma 2.6.6. *Let $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ and assume condition (P1) holds (i.e. $\lambda_\alpha \to 0$). Then, up to a subsequence,*

(a) $\nabla B_\alpha(x) \to 0$ pointwise almost everywhere on $\mathbb{R}^n$.

(b) $B_\alpha \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ and pointwise almost everywhere;

as $\alpha \to \infty$. Here, B_α are given by (2.6.13)

Proof. Fix $\varepsilon > 0$ and observe that, after a change of variables,

$$\int_{B(0,\varepsilon)^\complement} |\nabla B_\alpha(z)|^p \, |z|^{-ap} \, \mathrm{d}x = \int_{B\left(-x_0, \frac{\varepsilon}{\lambda_\alpha}\right)^\complement} |\nabla v\,(x)|^p \, |x + x_0|^{-ap} \, \mathrm{d}x.$$

The right-hand side of this equation converges to 0 by the monotone convergence theorem. Passing to a subsequence, we infer that $\nabla B_\alpha \to 0$ pointwise almost everywhere on $\mathbb{R}^n \setminus B(0, \varepsilon)$. As $\varepsilon > 0$ was arbitrary, a diagonal argument gives the existence of a subsequence such that $\nabla B_\alpha \to 0$ almost everywhere on $\mathbb{R}^n$.

Since (∇B_α) is bounded in $L^p_a(\mathbb{R}^n, 0)$, a simple change of variables and standard convergence results show that $B_\alpha \rightharpoonup 0$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$.

On the other hand, since (B_α) is bounded in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$, up to a subsequence, it converges both weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ and pointwise almost everywhere to some function in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$. By uniqueness of the weak limit, we conclude that $B_\alpha \to 0$ pointwise almost everywhere. □

Next, in Proposition 2.6.3, the *bubbles* take the form $\tau^-_{y_\alpha,\lambda_\alpha} v$ and we obtain a similar convergence result:

Lemma 2.6.7. *Let $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. If $a = b$ and condition (P1) holds (i.e. $\lambda_\alpha \to 0$) then, up to a subsequence,* *(a)* $\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right] \to 0$ *pointwise almost everywhere*

(b) $\tau^{-}_{y_\alpha,\lambda_\alpha} v \rightharpoonup 0$ *weakly in* $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ *and pointwise almost everywhere.*

Proof. Up to a subseqeunce, we may suppose that $y_\alpha \to y$. Given an arbitrary $\varepsilon > 0$, observe that for all α large

$$B\left(y_\alpha, \frac{\varepsilon}{2}\right) \subset B(y, \varepsilon).$$

Therefore,

$$\begin{aligned}\int_{B(y,\varepsilon)^\complement} \left|\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right]\right|^p |z|^{-ap}\, \mathrm{d}z &\le \int_{B(y_\alpha,\varepsilon/2)^\complement} \left|\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right]\right|^p |z|^{-ap}\, \mathrm{d}z \\ &\le C\left(\|v\|_{L^q\left(B(0,\varepsilon/(2\lambda_\alpha))^\complement\right)} + \|\nabla v\|_{L^p\left(B(0,\varepsilon/(2\lambda_\alpha))^\complement\right)}\right)^p\end{aligned}$$

where the second inequality follows from Remark 2.3.2 and $C > 0$ is a constant independent of α. Since $\lambda_\alpha \to 0$, the right-hand side tends to 0 by the dominated convergence theorem as $\alpha \to \infty$. Therefore,

$$\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right] \to 0 \qquad \text{in } L^p\left(B(y,\varepsilon)^\complement\right).$$

Since $\varepsilon > 0$ was arbitrary, a diagonal argument shows that, up to a subsequence, $\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right] \to 0$ almost everywhere.

By Lemma 2.3.2, the sequence $(\tau^{-}_{y_\alpha,\lambda_\alpha} v)$ is bounded in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$. Then, utilizing also that $\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right] \to 0$ pointwise almost everywhere, it follows from duality and a standard convergence result that $\nabla\left[\tau^{-}_{y_\alpha,\lambda_\alpha} v\right] \rightharpoonup 0$ weakly in $L^p_a(\mathbb{R}^n, 0)$. Put otherwise, $\tau^{-}_{y_\alpha,\lambda_\alpha} v \rightharpoonup 0$ weakly in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$.

On the other hand, since $\tau^{-}_{y_\alpha,\lambda_\alpha} v$ is a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$, up to a subsequence we may assume that there is some $w \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ such that

- $\tau^{-}_{y_\alpha,\lambda_\alpha} v \rightharpoonup w$ weakly in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$, and
- $\tau^{-}_{y_\alpha,\lambda_\alpha} v \to w$ pointwise almost everywhere.

Then, by uniqueness of weak limits, $w \equiv 0$. □

2.7 The Proof of Theorem 1.1.1

We now produce the proof of Theorem 1.1.1, i.e. we establish the main compactness result in the parameter case $a < b$. As previously stated, this setting can only produce weighted bubbles centered at the origin, which may also occur in the limit case $a = b$. We mention once more, in the spirit of transparency, that although this theorem is original research, it was first established during the author's master's degree in [12]. However, because the argument is essential in the more difficult $a = b \neq 0$ regime, it needs to be included somewhere within this thesis regardless. Choosing to include the proof in the $a < b$ case offers the best exposition possible; it allows for an organic treatment of the $a < b$ while highlighting the key distinctions between the distinct parameter cases and improving the concision of the proof of Theorem 1.1.2.

For the sake of brevity and clarity, we decompose the proof into several distinct steps.

Step 1. Being a Palais-Smale sequence, by Proposition 2.2.1, (u_α) is bounded in $\mathcal{D}_a^{1,p}(\Omega, 0)$. Hence, passing to a subsequence, u_α converges weakly to some $v_0 \in \mathcal{D}_a^{1,p}(\Omega, 0)$, with pointwise convergence almost everywhere on Ω, as $\alpha \to \infty$. Therefore, by Lemma 2.6.1, the sequence (u_α^1) defined by $u_\alpha^1 := u_\alpha - v_0$ satisfies

(i) $\|u_\alpha^1\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} + o(1)$;

(ii) $J(u_\alpha^1) \to c - J(v_0)$;

(iii) $J'(u_\alpha^1) \to 0$ in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$.

Furthermore, we find that $J'(v_0) = 0$ and $\nabla u_\alpha \to \nabla v_0$ a.e. on $\mathbb{R}^n$ as $\alpha \to \infty$. Especially, v_0 solves problem (A).

Step 2. If $u_\alpha^1 \to 0$ strongly in $L_b^q(\Omega, 0)$ then the proof is complete. Indeed,

$$\lim_{\alpha\to\infty} \int_\Omega \left(|x|^{-ap} \left|\nabla u_\alpha^1(x)\right|^p - |x|^{-bq} \left|u_\alpha^1(x)\right|^q\right) \mathrm{d}x = \lim_{\alpha\to\infty} \left|\left\langle J'(u_\alpha^1), u_\alpha^1\right\rangle\right| \to 0$$

since $J'(u_\alpha^1) \to 0$ in the dual of $\mathcal{D}_a^{1,p}(\Omega)$ and the sequence (u_α^1) is bounded in $\mathcal{D}_a^{1,p}(\Omega,0)$. Hence, $\nabla u_\alpha^1 \to 0$ strongly in $L_a^p(\Omega,0)$, i.e. $u_\alpha \to v_0$ strongly in $\mathcal{D}_a^{1,p}(\Omega,0)$. Thus, the conclusions of our theorem are satisfied for $k=0$ since

$$J(v_0) = \lim_{\alpha\to\infty} J(u_\alpha).$$

Step 3. If we do not satisfy the termination condition of Step 2, then, after passing to a subsequence,

$$\inf_{\alpha\geqslant 1} \int_\Omega \left|u_\alpha^1\right|^q |x|^{-bq}\,\mathrm{d}x > \delta$$

for some $\delta>0$. We may suppose without harm that

$$0 < \delta < \left(\frac{C_{a,b}}{2^p}\right)^{\frac{q}{q-p}}. \tag{2.7.1}$$

where $C_{a,b}>0$ is such that the CKN inequality (2.1.1) holds.

Consider the family $\{Q_\alpha\}_{\alpha\geqslant 1}$ of Lévy concentration functions each defined by

$$Q_\alpha(r) := \sup_{y\in\bar\Omega} \int_{B(y,r)} \left|u_\alpha^1\right|^q |x|^{-bq}\,\mathrm{d}x,$$

where each (u_α) is extended by 0 outside Ω. Note that, by CKN inequality (CKN), one has $Q_\alpha(r)<\infty$ for every $r\geq 0$. By continuity of each Q_α, we may apply the Intermediate Value Theorem to extract a smallest $\lambda_\alpha^1>0$ such that $Q_\alpha(\lambda_\alpha^1)=\delta$.

Finally, for each index α, we may select $y_\alpha^1 \in \overline{\Omega}$ such that

$$Q_\alpha(\lambda_\alpha^1) = \sup_{y\in\bar\Omega} \int_{B(y,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq}\,\mathrm{d}x = \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq}\,\mathrm{d}x = \delta. \tag{2.7.2}$$

Passing to a subsequence if necessary, we will assume that $\lambda_\alpha^1 \to \lambda^1 \geq 0$ and $y_\alpha^1 \to y^1 \in \overline{\Omega}$ as $\alpha\to\infty$.

Step 4. We show that the sequence of points $y_\alpha^1/\lambda_\alpha^1$ is bounded in $\mathbb{R}^n$. If not, passing to a subsequence we may assume that

$$\frac{|y_\alpha^1|}{\lambda_\alpha^1} \to \infty, \quad \text{as } \alpha \to \infty,$$

and that $|y_\alpha^1| > 4\lambda_\alpha^1 > 0$ for each $\alpha \in \mathbb{N}$. Consider the functions

$$v_\alpha(x) := \tau_{y_\alpha^1, \lambda_\alpha^1} u_\alpha^1(x) = \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^b \left(\lambda_\alpha^1\right)^\gamma u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)$$

with τ defined as in equation (2.3.3). Lemma 2.3.1 then implies that $v_\alpha \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ with $v_\alpha \to 0$ strongly. Furthermore, by the Gagliardo-Nirenberg-Sobolev inequality, we also obtain $v_\alpha \to 0$ in $L^{p^*}(\mathbb{R}^n)$. Using that $p^* \geq q$, Hölder's inequality further asserts that $v_\alpha \to 0$ in $L^q_{\text{loc}}(\mathbb{R}^n)$. However, Lemma 2.3.3 and (2.7.2) show that, as $\alpha \to \infty$,

$$\int_{B(0,1)} |v_\alpha(x)|^q \, dx = \int_{B(y_\alpha^1, \lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q |z|^{-bq} \, dz + o(1) = \delta + o(1),$$

contradicting our deduction that $v_\alpha \to 0$ strongly in $L^q_{\text{loc}}(\mathbb{R}^n)$.

Since $y_\alpha^1/\lambda_\alpha^1$ is bounded, passing to a subsequence, we may assume that

$$\frac{y_\alpha^1}{\lambda_\alpha^1} \to x_0 \in \mathbb{R}^n, \quad \text{as } \alpha \to \infty.$$

We then define a sequence of functions via homogeneity:

$$v_\alpha^1(x) = \left(\lambda_\alpha^1\right)^\gamma u_\alpha^1(\lambda_\alpha^1(x + x_0)).$$

It is clear that, for each $\alpha \in \mathbb{N}$, $v_\alpha^1 \in \mathcal{D}_a^{1,p}(\Omega_\alpha, -x_0) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ where

$$\Omega_\alpha := \frac{\Omega - \lambda_\alpha^1 x_0}{\lambda_\alpha^1} = \frac{\Omega}{\lambda_\alpha^1} - x_0.$$

By homogeneity, we mean that

$$\left\|v_\alpha^1\right\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)} = \|u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega, 0)}, \qquad \left\|v_\alpha^1\right\|_{L_b^q(\mathbb{R}^n, -x_0)} = \|u_\alpha\|_{L_b^q(\Omega, 0)}$$

In particular, (v^1_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)$ whence we may assume without loss of generality that

$$v^1_\alpha \rightharpoonup v^1 \text{ weakly in } \mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0), \quad v^1_\alpha \to v^1 \text{ a.e. in } \mathbb{R}^n$$

for some $v^1 \in \mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)$. We note that, by a change of variables

$$\sup_{y\in\overline{\Omega_\alpha}} \int_{B(y,1)} \left|v^1_\alpha(x)\right|^q |x+x_0|^{-bq}\,\mathrm{d}x = \sup_{w\in\overline{\Omega}} \int_{B(w,\lambda^1_\alpha)} \left|u^1_\alpha(z)\right|^q |z|^{-bq}\,\mathrm{d}z = \delta \tag{2.7.3}$$

where the last equality follows from equation (2.7.2).

Step 5. For a given $h \in \mathcal{D}^{1,p}_a(\Omega_\alpha,-x_0)$, define $h_\alpha \in \mathcal{D}^{1,p}_a(\Omega,0)$ via the homogeneous transform:

$$h_\alpha(z) := \left(\lambda^1_\alpha\right)^{-\gamma} h\left(\frac{z}{\lambda^1_\alpha} - x_0\right).$$

A change of variables shows that

$$\left\langle J'_{x_0,\infty}(v^1_\alpha), h\right\rangle = \left\langle J'(u^1_\alpha), h_\alpha\right\rangle.$$

Since $J'(u^1_\alpha)$ is a continuous linear functional on $\mathcal{D}^{1,p}_a(\Omega,0)$, for each $\alpha\in\mathbb{N}$, by a Riesz-type representation result, see (2.1.2), there exist functions $f^{(1)}_\alpha,\ldots,f^{(n)}_\alpha \in L^{p'}(\Omega,|x|^{-ap})$ so that

$$\left\langle J'(u^1_\alpha), h\right\rangle = \sum_{i=1}^n \int_\Omega f^{(i)}_\alpha(x)\partial_i h(x)\,|x|^{-ap}\,\mathrm{d}x, \quad \forall h\in\mathcal{D}^{1,p}_a(\Omega,0).$$

Of course, p' denotes the Hölder conjugate exponent to p. Moreover, because we have $J'(u^1_\alpha)\to 0$ in the strong operator topology,

$$\sum_{i=1}^n \int_\Omega \left|f^{(i)}_\alpha(x)\right|^{p'} |x|^{-ap}\,\mathrm{d}x \to 0, \quad \text{as } \alpha\to\infty. \tag{2.7.4}$$

Therefore, we may write

$$
\begin{aligned}
\left\langle J'_{x_0,\infty}(v^1_\alpha), h\right\rangle &= \left\langle J'(u^1_\alpha), h_\alpha\right\rangle \\
&= \sum_{i=1}^{n} \int_\Omega f^{(i)}_\alpha(z)\partial_i h_\alpha(z)\,|z|^{-ap}\,\mathrm{d}z \\
&= \left(\lambda^1_\alpha\right)^{n-ap-(\gamma+1)} \sum_{i=1}^{n} \int_{\Omega_\alpha} f^{(i)}_\alpha(\lambda^1_\alpha(x+x_0))\partial_i h\,(x)\,|x+x_0|^{-ap}\,\mathrm{d}x.
\end{aligned}
$$

It follows that

$$
\left\langle J_{x_0,\infty}(v^1_\alpha), h\right\rangle = \sum_{i=1}^{n} \int_{\Omega_\alpha} g^{(i)}_\alpha(x)\partial_i h\,(x)\,|x+x_0|^{-ap}\,\mathrm{d}x, \tag{2.7.5}
$$

where we have defined

$$
g^{(i)}_\alpha(x) := \left(\lambda^1_\alpha\right)^{\frac{n-ap}{p'}} f^{(i)}_\alpha\left(\lambda^1_\alpha(x+x_0)\right).
$$

Furthermore, by another change of variables,

$$
\sum_{i=1}^{n} \int_{\Omega_\alpha} \left|g^{(i)}_\alpha(x)\right|^{p'} |x+x_0|^{-ap}\,\mathrm{d}x = \sum_{i=1}^{n} \int_\Omega \left|f^{(i)}_\alpha(z)\right|^{p'} |z|^{-ap}\,\mathrm{d}z = o(1) \tag{2.7.6}
$$

as $\alpha \to \infty$ because of (2.7.4).

Step 6. By way of contradiction, we claim $v^1 \neq 0$. Assume $v^1 = 0$ so that $v^1_\alpha \rightharpoonup 0$ weakly in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ and pointwise a.e. on $\mathbb{R}^n$ as $\alpha \to \infty$. By the compactness of the embedding $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$, we may assume that $v^1_\alpha \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$ whence (v^1_α) is bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$. By Proposition 2.4.1, $\Omega_\alpha \to \Omega_\infty$ where

$$
\Omega_\infty := \begin{cases} \mathbb{R}^n & \text{if } \lambda^1 = 0, \\ \dfrac{\Omega}{\lambda^1} - x_0 & \text{if } \lambda^1 > 0, \end{cases}
$$

and let $\mathcal{C}$ be a countable cover of $\mathbb{R}^n$ such that if $\Lambda \in \mathcal{C}$ then either

1. $\Lambda = B(y, 1/2)$ for some $y \in \Omega_\infty$ or

2. Λ is compactly contained in the complement of $\overline{\Omega_\infty}$.

If $\Lambda \in \mathcal{C}$ is compactly contained in the complement of $\overline{\Omega_\infty}$, then Λ and Ω_α are disjoint for all α large since $\Omega_\alpha \to \Omega_\infty$. Therefore, v_α^1 vanishes on Λ for all α large and, in particular, $\nabla v_\alpha^1 \to 0$ in $L_a^p(\Lambda, -x_0)$.

On the other hand, suppose $\Lambda = B(y, 1/2)$ for some $y \in \Omega_\infty$. Since $\Omega_\alpha \to \Omega_\infty$, $y \in \Omega_\alpha$ for all α sufficiently large. Given an arbitrary $h \in C_c^\infty(\mathbb{R}^n; [0,1])$ with $\mathrm{supp}(h) \subset B(y, 1)$, using Hölder's inequality with the CKN-inequality (2.1.1) then yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} |h|^p \left|v_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x \\
&\quad= \int_{\mathrm{supp}(h)} \left|v_\alpha^1 h\right|^p \left|v_\alpha^1\right|^{q-p} |x+x_0|^{-bq} \,\mathrm{d}x \\
&\quad\leq \left(\int_{\mathrm{supp}(h)} \left|v_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x\right)^{\frac{q-p}{q}} \left(\int_{\mathrm{supp}(h)} \left|hv_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x\right)^{p/q} \\
&\quad\leq C_{a,b}^{-1} \left(\int_{\mathrm{supp}(h)} \left|v_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x\right)^{\frac{q-p}{q}} \left(\int_{\mathrm{supp}(h)} \left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap} \,\mathrm{d}x\right).
\end{aligned}$$

Then, bounding the first integral with equations (2.7.3) and (2.7.1),

$$\int_{\mathbb{R}^n} |h|^p \left|v_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x \leq 2^{-p} \int_{\mathrm{supp}(h)} \left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap} \,\mathrm{d}x. \tag{2.7.7}$$

On the other hand, since $v_\alpha^1 \to 0$ in $L_{\mathrm{loc}}^p(\mathbb{R}^n, |x+x_0|^{-ap})$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap} \,\mathrm{d}x &\leq 2^{p-1} \int_{\mathbb{R}^n} |h|^p \left|\nabla v_\alpha^1\right|^p |x+x_0|^{-ap} \,\mathrm{d}x + o(1) \\
&= 2^{p-1} \int_{\mathbb{R}^n} \left|\nabla v_\alpha^1\right|^{p-2} \left(\nabla v_\alpha^1, \nabla\left[|h|^p v_\alpha^1\right]\right) |x+x_0|^{-ap} \,\mathrm{d}x + o(1) \\
&= 2^{p-1} \left[\left\langle J_{x_0,\infty}'(v_\alpha^1), |h|^p v_\alpha^1 \right\rangle + \int_{\mathbb{R}^n} \left|v_\alpha^1\right|^q |h|^p |x+x_0|^{-bq}\right] + o(1).
\end{aligned}$$

Next, by Step 5 equations (2.7.5)-(2.7.6), we see that as $\alpha \to \infty$

$$\left\langle J'_{x_0,\infty}(v_\alpha^1), |h|^p\, v_\alpha^1 \right\rangle = \sum_{i=1}^n \int_{\Omega_\alpha} g_\alpha^{(i)} \partial_i \left(|h|^p\, v_\alpha^1 \right) |x + x_0|^{-ap}\, \mathrm{d}x = o(1)$$

where we have used that $(|h|^p\, v_\alpha^1)$ is bounded in $\mathcal{D}_a^{1,p}(\Omega_\alpha, -x_0)$. Ergo, concatenating with (2.7.7),

$$\int_{\mathbb{R}^n} \left|\nabla (h v_\alpha^1)\right|^p |x + x_0|^{-ap}\, \mathrm{d}x \le \frac{1}{2} \int_{\mathbb{R}^n} \left|\nabla (h v_\alpha^1)\right|^p |x + x_0|^{-ap}\, \mathrm{d}x + o(1).$$

We deduce that $\int_{\mathbb{R}^n} |\nabla (h v_\alpha^1)|^p\, |x + x_0|^{-ap}\, \mathrm{d}x \to 0$. Taking $h \equiv 1$ on $\Lambda = B(y, 1/2)$, we infer that $\nabla v_\alpha^1 \to 0$ in $L_a^p\left(\Lambda, -x_0\right)$. Having shown that this convergence holds for all $\Lambda \in \mathcal{C}$, it follows that $\nabla v_\alpha^1 \to 0$ in $L_{loc}^p(\mathbb{R}^n, |x + x_0|^{-ap})$. Multiplying with a cutoff function and applying the CKN-inequality, this implies that $v_\alpha^1 \to 0$ strongly in $L_{\mathrm{loc}}^q(\mathbb{R}^n, |x + x_0|^{-bq})$ as $\alpha \to \infty$. However, this contradicts by (2.7.2) since

$$\begin{aligned}
\int_{B(0,2)} \left|v_\alpha^1\right|^q |x + x_0|^{-bq}\, \mathrm{d}x &\ge \int_{B\left(\frac{y_\alpha^1}{\lambda_\alpha^1} - x_0, 1\right)} \left|v_\alpha^1\right|^q |x + x_0|^{-bq}\, \mathrm{d}x \\
&= \int_{B(y_\alpha^1, \lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q |z|^{-bq}\, \mathrm{d}x \\
&= \delta > 0
\end{aligned}$$

for all $\alpha \in \mathbb{N}$ sufficiently large.

Step 7. We claim that $\lambda^1 = \lim_{\alpha \to \infty} \lambda_\alpha^1 = 0$. To this end, we argue by contradiction and assume instead that $\lambda^1 > 0$. Given any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, a straightforward change of variables shows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\nabla v_\alpha^1(x), \varphi(x)\right) |x + x_0|^{-ap}\, \mathrm{d}x \\
&\quad = (\lambda_\alpha^1)^{(\gamma+1)-n+ap} \int_{\mathbb{R}^n} \left(\nabla u_\alpha^1(z), \varphi\left(\frac{z}{\lambda_\alpha^1} - x_0\right)\right) |z|^{-ap}\, \mathrm{d}z.
\end{aligned} \tag{2.7.8}$$

As the sequence (λ_α^1) is bounded, there is a compact set Λ such that, for each

$\alpha \in \mathbb{N}$, the function
$$z \mapsto \varphi\left(\frac{z}{\lambda_\alpha^1} - x_0\right)$$
is supported on Λ. Following this, since $\nabla u_\alpha^1 \to 0$ pointwise almost everywhere on $\mathbb{R}^n$ and is bounded in $L_a^p(\mathbb{R}^n, 0)$, standard convergence results imply that

$$\int_\Lambda \left|\nabla u_\alpha^1(z)\right| |z|^{-ap} \, \mathrm{d}z \to 0, \quad \text{as } \alpha \to \infty.$$

Using this and that $\lambda_\alpha^1 \to \lambda_1 > 0$, it follows from (2.7.8) that

$$\int_{\mathbb{R}^n} \left(\nabla v_\alpha^1(x), \varphi(x)\right) |x + x_0|^{-ap} \, \mathrm{d}x \to 0, \quad \text{as } \alpha \to \infty.$$

Since smooth functions of compact support are dense in $L^{p'}(\mathbb{R}^n, |x+x_0|^{-ap})$, it readily follows that $\nabla v_\alpha^1 \rightharpoonup 0$ weakly in $L_a^p(\mathbb{R}^n, -x_0)$. Put otherwise, $v_\alpha^1 \rightharpoonup 0$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ which directly contradicts Step 6.

Step 8. Since $v^1 \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ and $\Omega_\alpha \to \mathbb{R}^n$, there exists a sequence functions $\psi_\alpha \in C_c^\infty(\Omega_\alpha)$ such that $\psi_\alpha \to v^1$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ as $\alpha \to \infty$.

Consider the modified sequence given by a homogeneous transform:

$$\tilde{\psi}_\alpha(z) := \left(\lambda_\alpha^1\right)^{-\gamma} \psi_\alpha\left(\frac{z}{\lambda_\alpha^1} - x_0\right).$$

Since $\psi_\alpha \to v^1$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$, a change of variables (i.e. homogeneity) shows that
$$\tilde{\psi}_\alpha(\cdot) - \left(\lambda_\alpha^1\right)^{-\gamma} v^1\left(\frac{\cdot}{\lambda_\alpha^1} - x_0\right) \to 0 \quad \text{in } \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0).$$
Therefore, after an application of Lemma 2.6.6 (with $v = v^1$), we deduce that

(i) $\nabla \tilde{\psi}_\alpha \to 0$ pointwise almost everywhere on $\mathbb{R}^n$;

(ii) $\tilde{\psi}_\alpha \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\Omega, 0)$ and pointwise almost everywhere $\alpha \to \infty$.

Step 9. We outline the iteration process. Applying Proposition 2.6.2, we see that

v^1 solves a translated version of (B) and the sequence

$$u_\alpha^2(x) := u_\alpha^1(x) - \left(\lambda_\alpha^{(1)}\right)^{-\gamma} v^1\left(\frac{x}{\lambda_\alpha^{(1)}} - x_0^1\right),$$

for $\lambda_\alpha^{(1)} := \lambda_\alpha^1$, satisfies

$$\left\|u_n^2\right\|^p_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} = \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \left\|v^1\right\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n,-x_0^{(1)})} + o(1)$$

and

$$\begin{cases} J_{0,\infty}(u_\alpha^2) \to c - J(v_0) - J_{x_0^{(1)},\infty}(v^1), \\ J'_{0,\infty}(u_\alpha^2) \to 0 \text{ in } \mathcal{D}_a^{-1,p'}(\Omega,0). \end{cases}$$

Next, we consider the auxiliary sequence

$$\tilde{u}_\alpha^2(x) := u_\alpha^1(x) - \tilde{\psi}_\alpha(x)$$

which is bounded in $\mathcal{D}_a^{1,p}(\Omega,0)$. Because $\psi_\alpha \to v^1$ strongly in $\mathcal{D}^{1,p}(\mathbb{R}^n,-x_0^1)$,

$$\begin{cases} J\left(\tilde{u}_\alpha^2\right) \to c - J(v_0) - J_{x_0^{(1)},\infty}(v^1), \\ J'(\tilde{u}_\alpha^2) \to 0 \text{ in } \mathcal{D}_a^{-1,p'}(\Omega,0). \end{cases}$$

We may therefore apply Steps 2-9 to this new sequence $(\tilde{u}_\alpha^2)$, at each stage removing another solution v^j.

Step 10. For each index j, a suitable translation of v^j solves (B). Now, recalling Lemma 2.2.2, note that Palais-Smale sequences for problem (A) have non-negative limiting energy. Moreover, by Proposition 2.2.3, the energies of non-trivial critical points for the weighted limiting problem (B) are bounded below by a uniform positive constant. Thus, this procedure can only run its course finitely many times before we reach the termination condition dictated within Step 2. This completes the proof of Theorem 1.1.1.

2.8 The Proof of Theorem 1.1.2

We now begin the proof of Theorem 1.1.2, which contains the entire proof of Theorem 1.1.1 as a sub-argument. Naturally, the proof begins in the same way (first extracting a solution to (A) and recovering another (PS)-sequence by removing said solution). However, unlike Step 4 within the previous proof, we will not be able to assume $y_\alpha^1/\lambda_\alpha^1$ is bounded, because $\tau_{y_\alpha^1,\lambda_\alpha^1} u_\alpha^1$ seemingly does *not* necessarily converge strongly to 0 in all cases. In fact, we will see that in this situation, $\tau_{y_\alpha^1,\lambda_\alpha^1} u_\alpha^1$ may capture energy concentrating away from the origin, in such a manner that it does not "feel" the weights.

Steps 1-2-3. Steps 1,2 are identical to their respective counterparts from the proof in the $a < b$ case and we modify only the upper bound on $\delta > 0$ in step 3.

More precisely, as in step 1, we deduce that there is some $v_0 \in \mathcal{D}_a^{1,p}(\Omega, 0)$ such that, with $u_\alpha^1 := u_\alpha - v_0$, up to a subsequence,

(i) $u_\alpha \rightharpoonup v_0$ weakly in $\mathcal{D}_a^{1,p}(\Omega, 0)$ and pointwise almost everywhere;

(ii) $\|u_\alpha^1\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} + o(1)$;

(iii) $J(u_\alpha^1) \to c - J(v_0)$;

(iv) $J'(u_\alpha^1) \to 0$ in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$;

(v) $\nabla u_\alpha \to \nabla v_0$ and pointwise almost everywhere;

as $\alpha \to \infty$. Moreover $J'(v_0) = 0$ so v_0 solves problem (A). Then, as in step 2, the proof terminates if $u_\alpha^1 \to 0$ strongly in $L_b^q(\Omega, 0)$. Thus, we proceed to step 3 assuming that u_α^1 does not converge to 0 strongly in $L_b^q(\Omega, 0)$. In the latter case, up to a subsequence, we may suppose that for each index α there holds

$$\sup_{y\in\bar\Omega} \int_{B(y,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq} \,\mathrm{d}x = \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq} \,\mathrm{d}x = \delta. \tag{2.8.1}$$

for some $0 < \lambda_\alpha^1 \le \operatorname{diam}(\Omega)$ and $y_\alpha^1 \in \overline{\Omega}$. Here,

$$0 < \delta < \frac{1}{C_\tau^q} \left(\frac{C_{a,b}}{2^p}\right)^{\frac{q}{q-p}}, \tag{2.8.2}$$

where $C_{a,b} > 0$ is such that the CKN inequality (2.1.1) holds and $C_\tau > 0$ is the constant from Lemma 2.3.1. Finally, passing to another subsequence if necessary, we further assume that $\lambda_\alpha^1 \to \lambda^1 \geq 0$ and $y_\alpha^1 \to y^1 \in \overline{\Omega}$ as $\alpha \to \infty$.

Step 4. The following dichotomy holds: either $y_\alpha^1/\lambda_\alpha^1$ is bounded or unbounded. In the former case, we pass to a subsequence so that

$$\frac{y_\alpha^1}{\lambda_\alpha^1} \to x_0 \in \mathbb{R}^n$$

and proceed exactly as in the proof with $a < b$ (i.e. we follow Steps 5-10 from the previous section). Otherwise, we may pass to a subsequence in such a way that

$$\frac{|y_\alpha^1|}{\lambda_\alpha^1} \to \infty \quad \text{and} \quad |y_\alpha^1| > 4\lambda_\alpha^1.$$

In this case, it is automatic that $\lambda_\alpha^1 \to 0$ as $\alpha \to \infty$. Subsequently, we may define

$$v_\alpha^1 := \tau_{y_\alpha^1, \lambda_\alpha^1} u_\alpha^1 \in \mathcal{D}^{1,p}(\mathbb{R}^n).$$

Then, $v_\alpha^1 \in \mathcal{D}^{1,p}(\Omega_\alpha)$ in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ where

$$\Omega_\alpha := \frac{\Omega - y_\alpha^1}{\lambda_\alpha^1}$$

and, by Lemma 2.3.1, the sequence (v_α^1) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. Therefore, there exists $v^1 \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that

$$v_\alpha^1 \rightharpoonup v^1 \text{ weakly in } \mathcal{D}^{1,p}(\mathbb{R}^n), \quad v_\alpha^1 \to v^1 \text{ a.e. in } \mathbb{R}^n.$$

Furthermore, by Lemma 2.3.3 and (2.8.1),

$$\int_{B(0,1)} |v_\alpha^1|^q \, \mathrm{d}x = \delta + o(1) \qquad \text{as} \quad \alpha \to \infty.$$

On the other hand, by Remark 2.3.1 and (2.8.1), we see that

$$\sup_{y\in\overline{\Omega_\alpha}}\int_{B(y,1)}\left|v_\alpha^1\right|^q \mathrm{d}x \leq C_\tau^q \sup_{y\in\overline{\Omega}}\int_{B(y,\lambda_\alpha^1)}\left|u_\alpha^1\right|^q |z|^{-bq}\,\mathrm{d}x = C_\tau^q\delta$$

as $\alpha\to\infty$. Combining these last two equations, we obtain

$$\delta+o(1)\leq \sup_{y\in\overline{\Omega_\alpha}}\int_{B(y,1)}\left|v_\alpha^1\right|^q \mathrm{d}x \leq C_\tau^q\delta \leq \left(\frac{C_{a,b}}{2^p}\right)^{\frac{q}{q-p}} \tag{2.8.3}$$

where, for the last step, we used (2.8.2).

Step 5. Fix a compact set $K\subseteq\Omega$ and let (h_α) in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ be a bounded sequence with $h_\alpha\in\mathcal{D}^{1,p}(\Omega_\alpha)$ and h_α supported in K for each index $\alpha\in\mathbb{N}$. It follows from Lemma 2.5.1 and the CKN-inequality (CKN) that

$$\left\langle J_\infty'(v_\alpha^1),h_\alpha\right\rangle = \left\langle J'(u_\alpha^1),\tau^-_{y_\alpha^1,\lambda_\alpha^1}h_\alpha\right\rangle + o\left(1\right) \tag{2.8.4}$$

as $\alpha\to\infty$. By a Riesz-type representation result, i.e. characterization (2.1.2), we may find, for each α, functions $f_\alpha^{(1)},\ldots,f_\alpha^{(n)}$ in $L_a^{p'}(\Omega)$ such that

$$\left\langle J'(u_\alpha^1),h\right\rangle = \sum_{i=1}^n\int_\Omega f_\alpha^{(i)}(z)\partial_i h(z)\,|z|^{-ap}\,\mathrm{d}z,\qquad \forall h\in\mathcal{D}_a^{1,p}(\Omega,0).$$

Moreover, since $J'(u_\alpha^1)\to 0$ strongly in the dual of $\mathcal{D}_a^{1,p}(\Omega,0)$,

$$\sum_{i=1}^n\int_\Omega\left|f_\alpha^{(i)}(z)\right|^{p'}|z|^{-ap}\,\mathrm{d}z\to 0,\quad \text{as } \alpha\to\infty. \tag{2.8.5}$$

Combining (2.8.4) with the aforementioned functional representation, we may write

$$\begin{aligned}\left\langle J'_\infty(v^1_\alpha), h_\alpha\right\rangle &= \sum_{i=1}^n \int_\Omega f^{(i)}_\alpha(z)\partial_i\left[\tau^-_{y^1_\alpha,\lambda^1_\alpha} h_\alpha\right](z)\,|z|^{-ap}\,\mathrm{d}z + o(1)\\ &= \sum_{i=1}^n \int_\Omega f^{(i)}_\alpha(z)\partial_i\left[\tau^-_{y^1_\alpha,\lambda^1_\alpha} h_\alpha\right](z)\left[|z|^{-ap} - \left|y^1_\alpha\right|^{-ap}\right]\mathrm{d}z\\ &\quad + \sum_{i=1}^n \int_\Omega f^{(i)}_\alpha(z)\partial_i\left[\tau^-_{y^1_\alpha,\lambda^1_\alpha} h_\alpha\right](z)\left|y^1_\alpha\right|^{-ap}\mathrm{d}z + o(1).\end{aligned}$$

Since h_α is supported in K, each integrand is supported in $\lambda^1_\alpha K + y^1_\alpha$; then, using Lemma 2.3.4, we see that for all α large and each index $i = 1, \dots, n$

$$\begin{aligned}&\left|\int_\Omega f^{(i)}_\alpha(z)\partial_i\left[\tau^-_{y^1_\alpha,\lambda^1_\alpha} h_\alpha\right](z)\left[|z|^{-ap} - \left|y^1_\alpha\right|^{-ap}\right]\mathrm{d}z\right|\\ &\le C_K\frac{\lambda^1_\alpha}{|y^1_\alpha|}\int_\Omega \left|f^{(i)}_\alpha(z)\right|\left|\partial_i\left[\tau^-_{y^1_\alpha,\lambda^1_\alpha} h_\alpha\right](z)\right||z|^{-ap}\,\mathrm{d}z\\ &\le C_K\frac{\lambda^1_\alpha}{|y^1_\alpha|}\left\|f^{(i)}_\alpha\right\|_{L^{p'}_a(\Omega,0)}\|h_\alpha\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} = o(1)\end{aligned}$$

where we have used Hölder's inequality, Lemma 2.3.2 and equation (2.8.5). Returning to our earlier expression, we have

$$\begin{aligned}\left\langle J'_\infty(v^1_\alpha), h_\alpha\right\rangle &= \sum_{i=1}^n \int_\Omega f^{(i)}_\alpha(z)\partial_i\left[\tau^-_{y^1_\alpha,\lambda^1_\alpha} h_\alpha\right](z)\left|y^1_\alpha\right|^{-ap}\mathrm{d}z + o(1)\\ &= \left(\frac{\lambda^1_\alpha}{|y^1_\alpha|}\right)^{ap-2b}\left(\lambda^1_\alpha\right)^{n-2\gamma-1-ap}\int_{\Omega_\alpha}\left[\tau_{y^1_\alpha,\lambda^1_\alpha} f^{(i)}_\alpha\right]\partial_i h_\alpha \mathrm{d}x + o(1)\end{aligned}$$

Here, we have used that, since each h_α is supported in K, the cut-off η in the transform reduces to 1 on the integrand's support for all α large. Hence,

$$\left\langle J'_\infty(v^1_\alpha), h_\alpha\right\rangle = \sum_{i=1}^n \int_{\mathbb{R}^n} g^{(i)}_\alpha \partial_i h_\alpha \mathrm{d}x + o(1) \tag{2.8.6}$$

where, extending each $f_\alpha^{(i)}$ by 0 outside of Ω,

$$g_\alpha^{(i)} := \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{ap-2b} \left(\lambda_\alpha^1\right)^{n-2\gamma-1-ap} \tau_{y_\alpha^1,\lambda_\alpha^1} f_\alpha^{(i)}$$

Now, bounding as in Lemma 2.3.1, we see that

$$\int_{\mathbb{R}^n} \left|g_\alpha^{(i)}(x)\right|^{p'} \mathrm{d}x \le C \int_{\mathbb{R}^n} \left|f_\alpha^{(i)}(z)\right|^{p'} |z|^{-ap} \mathrm{d}z = o(1)$$

by (2.8.5). Therefore, applying Hölder's inequality to the right-hand side of equation (2.8.6) we conclude

$$\left\langle J'_\infty(v_\alpha^1), h_\alpha \right\rangle = o(1).$$

Step 6. We claim that $v^1 \neq 0$. By contradiction, suppose that $v^1 = 0$. Then, $v_\alpha^1 \rightharpoonup 0$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and $v_\alpha^1 \to 0$ pointwise almost everywhere. Furthermore, by the compactness of the embedding $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$, we may assume without loss of generality that $v_\alpha^1 \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Since $\lambda_\alpha^1 \to 0$, passing to another subsequence if necessary, it follows from Proposition 2.4.2 that there is a set $\Omega_\infty \subseteq \mathbb{R}^n$, which is either $\mathbb{R}^n$ or a half-space, such that

1. Any compact subset of Ω_∞ is contained in Ω_α for all α large.

2. Any compact subset of $\left(\overline{\Omega_\infty}\right)^{\complement}$ is contained in $\left(\overline{\Omega_\alpha}\right)^{\complement} \subseteq \Omega_\alpha^{\complement}$ for all α large.

We may cover $\mathbb{R}^n$ by a countable collection $\mathcal{C}$ of such that if $\Lambda \in \mathcal{C}$ then either

1. $\Lambda = B(y, 1/2)$ for some $y \in \Omega_\infty$ or

2. Λ is compactly contained in the complement of $\overline{\Omega_\infty}$.

Pick an arbitrary Λ in the collection $\mathcal{C}$. First, if Λ is compactly contained in the complement of $\overline{\Omega_\infty}$, then Λ is disjoint from $\overline{\Omega_\alpha}$ for all α large. Since v_α^1 is supported in Ω_α, it is clear that $\nabla v_\alpha^1 \to 0$ in $L^p(\Lambda)$.

On the other hand, if $\Lambda = B(y, 1/2)$ for some $y \in \Omega_\infty$ then, for all α large,

$y \in \Omega_\alpha$. Let $h \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\begin{cases} 0 \leq h \leq 1 & \\ h = 1 & \text{in } \Lambda = B(y, 1/2) \\ h = 0 & \text{outside } B(y,1). \end{cases}$$

By Hölder's inequality with exponents q/p and conjugate $q/(q-p) = n/p$ followed by the CKN inequality (CKN),

$$\begin{aligned} \int_{\mathbb{R}^n} |h|^p \left|v_\alpha^1\right|^q \mathrm{d}x &= \int_{\mathrm{supp}(h)} \left(\left|hv_\alpha^1\right|\right)^p \left|v_\alpha^1\right|^{q-p} \mathrm{d}x \\ &\leq \left(\int_{\mathrm{supp}(h)} \left|hv_\alpha^1\right|^q \mathrm{d}x\right)^{p/q} \left(\int_{\mathrm{supp}(h)} \left|v_\alpha^1\right|^q \mathrm{d}x\right)^{p/n} \\ &\leq C_{a,b}^{-1} \left(\int_{\mathrm{supp}(h)} \left|\nabla\left(hv_\alpha^1\right)\right|^p \mathrm{d}x\right) \left(\int_{B(y,1)} \left|v_\alpha^1\right|^q \mathrm{d}x\right)^{p/n}. \end{aligned}$$

Then, using equation (2.8.3), we deduce that for all α large (so that $y \in \Omega_\alpha$)

$$\begin{aligned} \int_{\mathbb{R}^n} |h|^p \left|v_\alpha^1\right|^q \mathrm{d}x &\leq \left(\left(\frac{C_{a,b}}{2^p}\right)^{\frac{q}{q-p}}\right)^{p/n} C_{a,b}^{-1} \int_{\mathbb{R}^n} \left|\nabla\left(hv_\alpha^1\right)\right|^p \mathrm{d}x \\ &= 2^{-p} \int_{\mathbb{R}^n} \left|\nabla\left(hv_\alpha^1\right)\right|^p \mathrm{d}x. \end{aligned} \tag{2.8.7}$$

On the other hand, using that $v_\alpha^1 \to 0$ in $L^p_{loc}(\mathbb{R}^n)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\nabla\left(hv_\alpha^1\right)\right|^p \mathrm{d}x &\leq 2^{p-1} \int_{\mathbb{R}^n} |h|^p \left|\nabla v_\alpha^1\right|^p \mathrm{d}x + o(1) \\ &= 2^{p-1} \int_{\mathbb{R}^n} \left|\nabla v_\alpha^1\right|^{p-2} \left(\nabla v_\alpha^1, \nabla\left[|h|^p v_\alpha^1\right]\right) \mathrm{d}x + o(1) \\ &= 2^{p-1} \left[\left\langle J_\infty'(v_\alpha^1), |h|^p v_\alpha^1\right\rangle + \int_{\mathbb{R}^n} \left|v_\alpha^1\right|^q |h|^p \mathrm{d}x\right] + o(1) \end{aligned}$$

as $\alpha \to \infty$. Then, by step 5, as $\alpha \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\nabla \left(hv_\alpha^1\right)\right|^p \mathrm{d}x &= 2^{p-1} \int_{\mathbb{R}^n} \left|v_\alpha^1\right|^q |h|^p \mathrm{d}x + o(1) \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} \left|\nabla \left(hv_\alpha^1\right)\right|^p \mathrm{d}x + o(1)\end{aligned}$$

where we have used equation (2.8.7). Ergo, $\int_{\mathbb{R}^n} |\nabla (hv_\alpha^1)|^p \,\mathrm{d}x \to 0$ which implies that $\nabla v_\alpha^1 \to 0$ in $L^p(\Lambda)$.

Having concluded that $\nabla v_\alpha^1 \to 0$ in $L^p(\Lambda)$ for any $\Lambda \in \mathcal{C}$, we deduce that $\nabla v_\alpha^1 \to 0$ in $L^p_{loc}(\mathbb{R}^n)$. Multiplying with a cutoff function and applying the CKN inequality (CKN), this clearly implies that $v_\alpha^1 \to 0$ in $L^q_{loc}(\mathbb{R}^n)$; this contradicts equation (2.8.3).

Step 7. In contrast with the previous case $a < b$, it is already known that $\lambda^1 = 0$. Consequently, Step 7 is superfluous here.

Step 8. Since $\Omega_\alpha \to \Omega_\infty$ and $v^1 \in \mathcal{D}^{1,p}(\Omega_\infty)$, there exist a sequence of functions (ψ_α) in $C_c^\infty(\Omega_\infty)$ with $\psi_\alpha \in C_c^\infty(\Omega_\alpha)$ for each α such that $\psi_\alpha \to v^1$ in $\mathcal{D}^{1,p}(\Omega_\infty) \subseteq \mathcal{D}^{1,p}(\mathbb{R}^n)$.

By Lemma 2.3.2, the sequence given by $\tilde{\psi}_\alpha := \tau^-_{y_\alpha^1,\lambda_\alpha^1} \psi_\alpha \in C_c^\infty(\Omega)$ satisfies

$$\left\|\tilde{\psi}_\alpha - \tau^-_{y_\alpha^1,\lambda_\alpha^1} v^1\right\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} = \left\|\tau^-_{y_\alpha^1,\lambda_\alpha^1}\left(\psi_\alpha - v^1\right)\right\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} \leq C \left\|\psi_\alpha - v^1\right\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)}$$

for some constant $C > 0$. Then, since $\psi_\alpha \to v^1$ in $\mathcal{D}^{1,p}(\mathbb{R}^n)$,

$$\tilde{\psi}_\alpha - \tau^-_{y_\alpha^1,\lambda_\alpha^1} v^1 \to 0 \quad \text{in } \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0) \tag{2.8.8}$$

Recalling Lemma 2.6.7, it readily follows that, up to a subsequence,

1. $\nabla\tilde{\psi}_\alpha \to 0$ pointwise almost everywhere;

2. The sequence $\tilde{\psi}_\alpha \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\Omega, 0)$ and pointwise almost everywhere.

Step 9. We now show how to iterate to the next stage. Define

$$u_\alpha^2 := u_\alpha^1 - \tau^-_{y_\alpha^1,\lambda_\alpha^1} v^1$$

By Proposition 2.6.3, $J'_\infty(v^1) = 0$ in Ω_∞ and, as $\alpha \to \infty$, we have

$$\|u_\alpha^2\|^p_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} = \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|v^1\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} + o(1)$$

and

$$\begin{cases} J_{0,\infty}(u_\alpha^2) \to c - J(v_0) - J_\infty(v^1) \\ J'_{0,\infty}(u_\alpha^2) \to 0 & \text{strongly in the dual of } \mathcal{D}_a^{1,p}(\Omega,0). \end{cases}$$

Next, consider the auxillary sequence

$$\tilde{u}_\alpha^2 := u_\alpha^1 - \tilde{\psi}_\alpha$$

which is bounded in $\mathcal{D}_a^{1,p}(\Omega,0)$. Since, by Step 8,

$$u_\alpha^2 - \tilde{u}_\alpha^2 = \tilde{\psi}_\alpha - \tau^-_{y_\alpha^1,\lambda_\alpha^1} v^1 \to 0 \quad \text{strongly in } \mathcal{D}_a^{1,p}(\mathbb{R}^n,0)$$

we see that

$$\begin{cases} J(\tilde{u}_\alpha^2) \to c - J(v_0) - J_\infty(v^1) \\ J'(\tilde{u}_\alpha^2) \to 0 & \text{strongly in the dual of } \mathcal{D}_a^{1,p}(\Omega,0). \end{cases}$$

We may therefore apply steps 2-9 to this new sequence $(\tilde{u}_\alpha^2)$, at each stage removing another solution v^j.

Step 10. At the j^{th} iteration stage, we had one of the following cases.

1. If $y_\alpha^j/\lambda_\alpha^j$ is bounded then, as in the $a < b$ case, a suitable translation of v^j solves (B).
2. If the ratio $y_\alpha^j/\lambda_\alpha^j$ is unbounded, then $J'_\infty(v^j) = 0$ in a set Ω_∞. Here, either
   - $\Omega_\infty = \mathbb{R}^n$ and v^j solves (C).

- Ω_∞ is a half-space, so a translated rotation of v^j solves the limiting problem (D) in $\mathbb{H} = \{x \in \mathbb{R}^n : x_n > 0\}$.

The procedure terminates because, by Lemma 2.2.2, Palais-Smale sequences necessarily possess non-negative limiting energy and, by Proposition 2.2.3, at each iteration stage, we remove a solution whose energy is uniformly strictly positive.

Chapter 3

Existence of Entire Nodal Solutions and Symmetry Prescription

The purpose of this chapter is to leverage the previously developed representation theorems in order to answer the question of nodal existence for solutions to the critical weighted problem in the whole of $\mathbb{R}^n$. As mentioned in Chapter 1, the motivation for this endeavour is substantial. Primarily, we are interested in extending the known nodal existence theory for the unweighted critical p-Laplace equation due to Clapp-Rios [24] and Clapp-Vicente-Benítez [21]. Within the first aforementioned paper, the authors give what they state to be the first exhibition of nodal solutions for the critical p-Laplace problem (1.1.6). Moreover, recently in 2025, Clapp-Vicente-Benítez [21] extended this to demonstrate that one can construct solutions for an infinite number of *distinct* symmetry configurations. We note, however, that these symmetries are restricted to 4-real coordinates, and to isomorphic copies of said symmetries.

Aside from the paper of Szulkin-Willem [64], which establishes sign-changing solutions in the $0 \leq a < b$ by using a truncated radial minimizer for the Caffarelli-Kohn-Nirenberg *inequality* and a concentration compactness principle (following the spirit of P.L. Lions [48]-[49]), the literature on nodal solutions to the pure critical Caffarelli-Kohn-Nirenberg problem in $\mathbb{R}^n$ is extremely sparse. This is especially true in the $a < 0$ where, to the best of the author's knowledge, no sign-changing solutions have been found when $p \neq 2$. However, by virtue of the global compactness theory developed in Chapter 2, we *are* in a comfortable position for resolving this gap. In fact, we shall demonstrate that the pure critical Caffarelli-Kohn-Nirenberg problem in $\mathbb{R}^n$ admits non-trivial sign-changing solutions, for an infinity of mutually exclusive symmetry configurations, for all dimensions $n \geq 4$ if $a < b$, and all dimensions $n \geq 4$ with $n \neq 5$ is $a = b \neq 0$. Moreover, we widen the class of possible symmetries one can prescribe to these solutions and, in particular, find new sign-changing solutions to both the unweighted critical p-Laplace problem (1.1.5), and the Yamabe problem (1.0.5).

For the sake of clarity and future reference within this chapter, we restate the problem of interest:

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bq}|u|^{q-2}u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0) \end{cases} \tag{B}$$

We recall as well that this is the Euler-Lagrange problem associated to the Caffarelli-Kohn-Nirenberg inequality [9], and is described by the energy functional

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p |x|^{-ap}\,\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^n} |u|^q |x|^{-bq}\,\mathrm{d}x. \tag{3.0.1}$$

That is, a function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ is a (weak) solution to (B) provided the (Fréchet) derivative $J'(u)$ vanishes, i.e. if $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}_a^{1,p}(\mathbb{R}^n)$ where $\langle \cdot, * \rangle$ denotes the duality pairing. Of course, all parameters are subject to the constraint dictated by (1.0.1). Henceforth, we work in dimension $n \geq 4$.

In what follows, we employ an approach relying upon the imposition of carefully chosen symmetries, described concretely by the appropriate subgroup of $O(n)$ and a corresponding group homomorphism. By studying the problem (B) restricted to functions satisfying these symmetries, we recover sufficient control over the Struwe profile of Palais-Smale sequences, within this symmetrized framework, to obtain a sign-changing solution to (B) in the limit. What is more, this function will necessarily respect the symmetries imposed on the Palais-Smale sequence, as a by product of the method. This naturally raises the question of whether one may systematically construct these groups G in such a way that we may distinguish the resulting solutions by their inherent symmetries.

The structure of this chapter is as follows. First, we describe a framework allowing for the construction of solutions to (B) in an abstract sense. More precisely, given a subgroup G of $O(n)$ and a homomorphism $\phi_G : G \to \{\pm 1\}$, satisfying certain requirements, we prove a result which ensures the existence of an associated nodal solution to (B) respecting the symmetries described by (G, ϕ_G). We mention that, as a by product of the distinction between the unbalanced case $a < b$ and the balanced case $a = b \neq 0$, there are substantial technical difficulties to overcome in the latter case. After having given general existence conditions, we address the question of symmetry categories previously mentioned. That is, we develop a systematic category of symmetries one may prescribe to sign-changing solutions of (B), and give conditions under which these symmetries are incompatible, i.e. describe conditions where we can ensure the symmetry types are mutually exclusive. This endeavour serves to both ensure a "variety" of possible nodal solutions, while also expanding the class of symmetries imposed by earlier constructions (e.g. Clapp-Rios [24] and Clapp-Vicente-Benítez [21]).

3.1 Symmetry Imprints

In this section, we develop the framework for our group symmetries, following Clapp-Rios [24] and Clapp-Benítez [21]. Throughout this chapter, unless otherwise stated, we fix a dimension $n \geq 4$ and denote by $O(n)$ the orthogonal group on $\mathbb{R}^n$. In addition, G is always assumed to be a *closed* subgroup of $O(n)$. Given a point $x \in \mathbb{R}^n$, we write $\mathrm{Orb}_G(x)$ to denote the orbit of x under the action of G, and $\mathrm{Stab}_G(x)$ to denote the stabilizer of x. A set $\Omega \subseteq \mathbb{R}^n$ is said to be G-invariant if

$$\mathrm{Orb}_G(\Omega) := \bigcup_{g \in G} \mathrm{Orb}_G(x) = \Omega.$$

We remark that the unit ball in $\mathbb{R}^n$ is G-invariant, no matter the subgroup G being considered. If $\Omega \subseteq \mathbb{R}^n$ is G-invariant, a function $u : \Omega \to \mathbb{R}$ is called G-invariant provided

$$u(gx) = u(x), \quad \forall x \in \Omega,\ g \in G.$$

Developing more notation, we will write ϕ_G to describe a continuous group homomorphism

$$\phi_G : G \to \{\pm 1\}.$$

Under these conventions, a pair (G, ϕ_G) shall be called a *symmetry imprint.* This choice of terminology will become clearer after the following definition, in which we declare a class of Sobolev functions respecting the structure imposed by the pair (G, ϕ_G). First however, we give a pointwise definition of ϕ_G-equivariance, which has been historically used in to address questions of existence and symmetry prescription (for example, Clapp-Rios [24]).

Definition 3.1.1. Let (G, ϕ_G) be a symmetry imprint and $\Omega \subseteq \mathbb{R}^n$ be G-invariant. A function $u : \Omega \to \mathbb{R}$ is called ϕ_G-equivariant on Ω if there holds

$$u(gx) = \phi_G(g)u(x)$$

for all $x \in \Omega$ and every $g \in G$.

Loosely speaking, by considering a ϕ_G-equivariant function, we are imposing a minimum of symmetry upon the functions being treated. Conveniently, we denote by $C_c^\infty(\Omega)^{\phi_G}$ the real vector space consisting of all real valued smooth functions, with compact support in Ω, which are ϕ_G-equivariant. To ensure that this space is meaningful, we impose the following two crucial conditions upon the symmetry imprint (G, ϕ_G).

(P1) There exists $\xi \in \mathbb{R}^n$ such that $\mathrm{Stab}_G(\xi) \subseteq \ker \phi_G$.

(P2) ϕ_G is a surjection.

We observe that these two properties are not original to this thesis. In fact, these are the same conditions used by Clapp-Rios [24]. Property (P1) is crucial to this approach as it ensures that the space of ϕ_G-equivariant test functions $C_c^\infty(\Omega)^{\phi_G}$ is non-trivial, and hence infinite dimensional (we refer to the work of Bracho-Clapp-Marzantowicz [6, page 195]). Of course, property (P2) ensures that any non-trivial

ϕ_G-equivariant function be sign-changing. This is of importance in our case as we are mainly interested in the existence of sign-changing solutions.

Remark 3.1.1 (On the stabilizer condition). As stated above, condition (P1) cannot be dispensed with. In fact, if (P1) fails and (P2) holds, then the trivial function is the only ϕ_G-equivariant function. What is more, pairs (G, ϕ) satisfying (P2) but not (P1) do exist. For example, for all dimensions $n \geq 2$, consider $G := O(n)$ with $\phi_G(g) := \det g$. Obviously, ϕ_G is a continuous and surjective homomorphism of groups. However, condition (P1) fails.

To see this, first observe that $\ker \phi_G = SO(n) \subset G$. As $\mathrm{Stab}_G(0) = O(n)$ we see that $\xi = 0$ fails the condition in (P1). For $\xi \neq 0$, write $\mathbb{R}^n = H \oplus H^\perp$ where $H \subset \mathbb{R}^n$ is a hyperplane of dimension $(n-1)$, passing through the origin, containing ξ. Define g_H to be the element of $O(n)$ which reflects across the hyperplane H. Clearly, $g_H \in \mathrm{Stab}_G(\xi)$ while $\phi_G(g_H) = \det(g_H) = -1$ ensures that $g_H \notin \ker \phi_G$.

As mentioned previously, before building the category of $(\alpha, \mathfrak{m})$ symmetries which are the centerpiece of Theorem 1.2.1, we first state a plain existence theorem which guarantees the existence of *a* solution with ϕ_G-equivariance if the symmetry imprint (G, ϕ_G) is appropriate. As one naturally ascertains from the Struwe-type decomposition in Chapter 2, the situation is much more relaxed in the $a < b$ setting. Naturallly, this is subject of the first main result of this section:

Theorem 3.1.1. *Let $n \geq 2$ and $a < b$ in* (B)*, Suppose that (G, ϕ_G) is a symmetry imprint satisfying (P1)-(P2). Then, there exists a non-trivial sign-changing $C^1(\mathbb{R}^n \setminus \{0\})$-solution to* (B) *which is ϕ_G-equivariant on $\mathbb{R}^n$.*

Remark 3.1.2. The C^1-regularity of this solution is not a consequence of this theorem and is well known for finite energy solutions of (B) (see Shakerian-Vétois [60]). However, it is mentioned because it means that u is defined pointwise away from the origin, ensuring that the ϕ_G-equivariance of u holds in a manner which is both meaningful and consistent with Definition 3.1.1.

An elegant facet of this theorem, is how easy it is to construct symmetry imprints (G, ϕ_G) which satisfy conditions (P1)-(P2). The reason these two conditions are sufficient to guarantee a solution with corresponding symmetries is precisely a consequence of the simplified bubbling phenomena created in the $a < b$ regime.

More precisely, it is due to the fact that, when $a < b$, any loss of compactness necessarily takes place at the origin and leads to a solution to the weighted problem. We help to illustrate this with a simple example, tackling straightforward finite group symmetry prescription.

Example 3.1.1. Fix a dimension $n \geq 2$ and let $S \subseteq \{1, \ldots, n\}$ be non-empty. For each index $j \in S$, let ρ_j be the linear isometry of $\mathbb{R}^n$ given by

$$\rho_j(x_1, \ldots x_{j-1}, x_j, x_{j+1}, , x_n) := (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n).$$

Define G to be the subgroup of $O(n)$ generated[1] by $\{\rho_j\}_{j \in S}$. and let $\phi_G : G \to \{\pm 1\}$ be the surjective homomorphism induced by the relation $\phi_G(\rho_j) = -1$, for each $j \in S$. Clearly, being that $G \subset O(n)$ is finite and therefore inherits the discrete topology, this group homomorphism is continuous. Furthermore, it is obvious that $\xi = (1, \ldots, 1)$ is a point for which (P1) holds true. Consequently, (G, ϕ_G) is a symmetry imprint satisfying conditions (P1)-(P2) and Theorem 3.1.1 ensures the existence of a non-trivial solution u_S to (B) which is ϕ_G-equivariant, i.e. a $C^1(\mathbb{R}^n \setminus \{0\})$-solution such that

$$u(\rho_j(x)) = -u(x),$$

for every $x \in \mathbb{R}^n \setminus \{0\}$ and every $j \in S$.

In fact, as pointed out by Professor Clapp, this gives rise to the existence of nodal solutions for an infinite number of distinctive symmetry-types even in the low dimensions $n = 2$ and $n = 3$. This is illustrated in the following example.

Example 3.1.2. Let $n \geq 2$ and write $\mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2}$ with $x = (z, y)$ where $x \in \mathbb{R}^n$, $z \in \mathbb{C}$, $y \in \mathbb{R}^{n-2}$, and define for any $m \geq 0$ the linear isometry

$$\rho_m : \mathbb{R}^n \to \mathbb{R}^n, \quad (z, y) \mapsto \left(e^{i\frac{\pi}{2^m}}, y\right).$$

Let $\Lambda_m \cong O(n-2)$ denote the copy of the orthogonal group acting on the last $(n-2)$-coordinates of $\mathbb{R}^n$, according to our representation above. Declare G_m to

[1] This group is finite and merely consists of words of the form $\prod_{j \in S} \rho_j^{\varepsilon_j}$ with $\varepsilon_j \in \{0, 1\}$

be the subgroup of $O(n)$ generated by $\{\rho_m\} \cup \Lambda_m$ and observe that ρ_m has order $2^{(m+1)}$. Consequently, G_m is a compact subgroup of $O(n)$ and the surjection

$$\phi_{G_m} :\to \{\pm 1\}, \quad \phi_{G_m}(\rho_m^j) = (-1)^j,$$

is a well defined homomorphism of groups. Moreover, notice that the pair (G_m, ϕ_{G_m}) satisfies (P1) for any $\xi = (z, y) \in \mathbb{R}^n$, where $\xi = (z, y)$ with $|z| = 1$. Thus, (G_m, ϕ_m) is a symmetry imprint and Theorem 3.1.1 ensures the existence of an associated non-trivial nodal solution u_m which is ϕ_m-equivariant. What is more, if $m \neq k$ then $u_m \neq u_k$. Indeed, suppose $0 \leq m < k$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be simultaneously (G_m, ϕ_m)-equivariant and (G_k, ϕ_k)-equivariant. Then, noticing that $\rho_m = \rho_k^{2^{k-m}}$ we find that for each $x \in \mathbb{R}^n$ one has

$$u(x) = u(\rho_k^{2^{k-m}}(x)) = u(\rho_m(x)) = -u(x).$$

Hence, u vanishes identically. This implies that $u_m \neq u_k$. Finally, we note that for all dimensions $n \geq 4$ these groups G_m are infinite, unlike those built in Example 3.1.1. However, the group G_m does not have infinite orbit everywhere away from the origin. For example, the point $x = (1, \mathbf{0}) \in \mathbb{C} \times \mathbb{R}^{n-2}$ has finite orbit as ρ_m has finite order.

We now turn our attention towards the limit case $a = b \neq 0$ in the problem parameters which, as seen in the previous chapter, is significantly more complex that the case $a < b$ or $a = b = 0$. In fact, because of the distinctive bubbling phenomena we encounter in this case (most notably the possible loss of compactness in the form of a solution to the unweighted problem (C)), properties (P1)-(P2) on their own are not sufficient to guarantee the existence of a non-trivial sign-changing solution to (B). In this setting, we require the following additional constraint on our group G.

(P3) For each $x \in \mathbb{R}^n \setminus \{0\}$, the orbit $\mathrm{Orb}_G(x)$ is infinite.

Loosely speaking, condition (P3) ensures that the only loss of compactness possible from a suitable Palais-Smale sequence takes place at the origin, at sufficient rate. More precisely, if a Palais-Smale sequence were to exhibit concentration phenomena producing solutions to the unweighted problem (and therefore away from the origin), then the infinite orbit condition would lead to concentration along an infinite orbit, contradicting the boundedness of the Palais-Smale sequence. Furthermore, this reasoning explains why this condition may be omitted in the unbalanced case $a < b$, since all bubbling is centered at the origin in this case by Theorem 1.1.1. Condition (P3) is a subtle, and yet important, distinction from that present in Clapp-Rios [24], designed precisely to handle the distinctive concentration phenomena occuring for $a = b \neq 0$. Such a condition is *not* necessary for the unweighted model case $a = b = 0$ treated by Clapp-Rios [24] (see also Clapp-Vicente-Benítez [21]) because the problem (C) is inherently translation invariant due to the lack of weights. Indeed, if $a = b = 0$, it is enough for the subgroup G to satisfy the weaker condition:

(P4) For each $x \in \mathbb{R}^n$ there holds $\dim(\mathrm{Orb}_G(x)) > 0$ or $\mathrm{Orb}_G(x) = \{x\}$.

Obviously, (P3) implies (P4) but the converse fails. We also observe that the group given in Example 3.1.1 fails both conditions (P3)-(P4). In fact, no finite group G can satisfy condition (P3). As for Example 3.1.2, we remark that the groups constructed therein are infinite for all $n \geq 4$, but these groups fail condition (P3). This supports an overarching theme of this thesis: the CKN problem falls into three distinctive qualitative categories: $a < b$, $a = b = 0$, and $a = b \neq 0$.

In light of our discussions and development, we are ready to state the general existence result for the $a = b \neq 0$ case. Our general existence result for the case $a = b \neq 0$ can now be stated.

Theorem 3.1.2. *Let $n \geq 2$ with $a = b \neq 0$, and suppose that (G, ϕ_G) is a symmetry imprint satisfying (P1)-(P2)-(P3). Then, there exists a non-trivial sign-changing $C^1(\mathbb{R}^n \setminus \{0\})$-solution to (B) which is ϕ_G-equivariant on $\mathbb{R}^n$.*

As before, we only mention that the solution is $C^1(\mathbb{R}^n \setminus \{0\})$ to emphasize that the function is pointwise defined, thereby giving meaning to the symmetry property.

We should also remark that we exclude the case $a = b = 0$ from the statement of this last theorem, since it is covered by Clapp-Rios [24, Theorem 3.1] which replaces condition (P3) with its weaker analogue (P4). Before we proceed further, we also note that for these two theorems to be meaningful, we must construct symmetry imprints which satisfy these conditions. Obviously, we have already done so within Example 3.1.1, but this only applies to the sub-limit case $a < b$ by our previous observations. In fact, as we mentioned earlier, it is natural in view of these theorems, to ask whether one can design a systematic category of distinct, mutually exclusive, symmetry types the solutions can satisfy. In other words, can we define a large class of symmetry types in which we can easily distinguish the solutions we find by their inherent symmetries. Put otherwise, can we also address questions of multiplicity by pushing further the notions of prescribed symmetry developed by authors such as Clapp-Rios [24] and Clapp-Vicente-Benítez [21].

The outline for this chapter will be as follows. First, we will develop the variational background for the problem (B). Especially, we will introduce a symmetrized version of the problem, which we shall use to extract a solution with the appropriate symmetries. To do so, we will use the Nehari Manifold method and the *principle of symmetric criticality*. More precisely, we use a mountain-pass argument to construct a symmetrized Palais-Smale sequence, operating at the appropriate "minimal energy" level. Then, we shall decompose this Palais-Smale sequence, by a global compactness analysis, to prove Theorem 3.1.1 and Theorem 3.1.2. Following this, after having given a general group theoretic framework under which an appropriate solution exists, we will precisely define the broad category of symmetries we consider. Here, we will construct the associated symmetry imprints and investigate their mutual incompatibility, thereby demonstrating that the resulting solutions to (B) possess distinct, mutually exclusive, symmetries. To rigorously investigate the incompatibility of these symmetry types, we will employ a binary coding scheme, together with a gcd-argument, giving rise to the surprising gcd-condition appearing in the statement of Theorem 1.2.1.

3.2 Equivariant Palais-Smale Sequences

Since our variational approach involves showing that the "right" Palais-Smale sequence necessarily leads to the desired solution of (B), it is key that we ensure such a sequence indeed exists. As such, the purpose of this section is to construct Palais-Smale sequences within the energy space, at an appropriate *minimal energy level*, satisfying the prescribed symmetry requirements. Subsequently, we shall conduct a compactness profile analysis on such a sequence, and show that it necessarily yields the existence of a non-trivial sign-changing solution to (B). Speaking informally, particularly in the $a = b \neq 0$ case, it is not enough that we restrict the energy of a Palais-Smale sequence. As we mentioned, when $a = b \neq 0$, such a sequence could potentially blow up away from the origin, producing a solution to the unweighted problem (C), which does not address the question of existence for (B). To correct this, we must choose our symmetries in such a way that they "cage" or restrict the blow-up configuration of our chosen sequence. Consequently, it is best if we build our Palais-Smale sequence itself from ϕ_G-equivariant functions within the space $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)^{\phi_G}$.

For the entirety of this section, we assume that (G, ϕ_G) is any symmetry imprint for which (P1) holds true. We emphasize that the results in this section do *not* require assumptions (P2) and (P3) for the pair (G, ϕ_G).

We define the ambient symmetric energy space $\mathcal{D}_a^{1,p}(\Omega, 0)^{\phi_G}$ to be the topological closure of $C_c^\infty(\Omega)^{\phi_G}$ within the weighted homogeneous Sobolev space $\mathcal{D}_a^{1,p}(\Omega, 0)$. Naturally, as a closed subspace of $\mathcal{D}_a^{1,p}(\Omega, 0)$, the symmetric space $\mathcal{D}_a^{1,p}(\Omega, 0)^{\phi_G}$ inherits the structure of a reflexive Banach space. Moreover, the elements of this space behave as expected with regards to the symmetries described by the imprint (G, ϕ_G). This basic property is verified by the next lemma.

Lemma 3.2.1. *Let $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)^{\phi_G}$. Then, after modification on a set of measure zero, there exists a set $E \subseteq \mathbb{R}^n$, of full Lebesgue measure, such that*

$$u(gx) = \phi_G(g)u(x)$$

for all $g \in G$ and all $x \in E$. In other words, $u(gx) = \phi_G(g)u(x)$ almost everywhere on $\mathbb{R}^n$, uniformly in $g \in G$. Furthermore, if u admits a Lebesgue representative in $C(\mathbb{R}^n \setminus \{0\})$, then this representative w satisfies

$$w(gx) = \phi_G(g)w(x)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and every $g \in G$.

Proof. For the duration of this proof, we forgo the usual abuse of notation and regard u not as an equivalence class of functions, but rather as any chosen representative of its Sobolev equivalence class.

By definition, there exists a sequence (φ_α) in $C_c^\infty(\mathbb{R}^n)^{\phi_G}$ which converges to u in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ as $\alpha \to \infty$. Passing to a subsequence if necessary, we may assume without harm that $\varphi_\alpha \to u$ almost everywhere on $\mathbb{R}^n$, i.e. there exists a set $F \subseteq \mathbb{R}^n$ of full measure such that $\varphi_\alpha(x) \to u(x)$, as $\alpha \to \infty$, for all $x \in F$. Define

$$v(x) := \lim_{\alpha\to\infty} \varphi_\alpha(x),$$

and let $E \subseteq \mathbb{R}^n$ denote the set of all points where v is defined. Now, since $E \supseteq F$, it is clear that E has full measure and that $v = u$ almost everwhere on $\mathbb{R}^n$. Moreover, if $x \in E$, then because $\varphi_\alpha \in C_c^\infty(\mathbb{R}^n)^{\phi_G}$ for every $\alpha \in \mathbb{N}$, we see that

$$\lim_{\alpha\to\infty} \varphi_\alpha(gx) = \lim_{\alpha\to\infty} \phi_G(g)\varphi_\alpha(x) = \phi_G(g) \lim_{\alpha\to\infty} \varphi_\alpha(x) = \phi_G(g)v(x),$$

whence $gx \in E$ and $v(gx) = \phi_G(g)v(x)$ by definition of v. This verifies the first claim, i.e. v is a Lebesgue representative of u for which $v(gx) = \phi_G(g)v(x)$ for all $g \in G$ and $x \in E$. For the remainder of the proof, we replace u by this better representative v.

Next, suppose that w is a $C(\mathbb{R}^n \setminus \{0\})$ representative of u. Hence, $w = v$ almost everywhere on $\mathbb{R}^n$ and so, we can find a set $E' \subseteq \mathbb{R}^n$ of full measure such that $w(x) = v(x)$ for all $x \in E'$. Fix $g \in G$, define $E^* := E \cap E' \cap g^{-1}E'$, and observe that E^* has full measure in $\mathbb{R}^n$ because g^{-1} is a linear isometry of $\mathbb{R}^n$. Thus, E^* is dense in $\mathbb{R}^n$ whence, given a point $x \in \mathbb{R}^n \setminus \{0\}$, we may select a sequence (x_k) in E^* such that $x_k \to x$ as $k \to \infty$. Then, it follows by the continuity of w that

$$\begin{aligned} w(gx) = \lim_{k\to\infty} w(gx_k) = \lim_{k\to\infty} v(gx_k) &= \lim_{k\to\infty} \phi_G(g)v(x_k) \\ &= \phi_G(g) \lim_{k\to\infty} v(x_k) \\ &= \phi_G(g) \lim_{k\to\infty} w(x_k) \\ &= \phi_G(g)w(x). \end{aligned}$$

This ends the proof. □

A key detail in the proofs of Theorems 3.1.1 and 3.1.2 is the restriction of problem (B) to a smaller subspace, consisting of Sobolev functions with sufficient symmetry. Therefore, as a natural "stepping stone" in the direction of full solutions, we consider solutions to a "symmetrized" version of problem (B), determined by testing against a smaller class of test functions that are ϕ_G-equivariant. Formally, given a symmetry imprint (G, ϕ_G) satisfying (P1) and a G-invariant domain $\Omega \subseteq \mathbb{R}^n$, we shall say that $u \in \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$ is a solution to

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = |x|^{-bq}\,|u|^{q-2}\,u \quad \text{in } \Omega \\ u \in \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G} \end{cases} \tag{3.2.1}$$

provided $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$. We also point out that, after an extension by 0 outside Ω, it is automatic that

$$\mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G} \subseteq \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)^{\phi_G} \subseteq \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0).$$

It will also become convenient to explicitly introduce an analogue of this symmetrized problem, in the whole of $\mathbb{R}^n$. That is, a function $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)^{\phi_G}$ is called a solution to

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = |x|^{-bq}\,|u|^{q-2}\,u \quad \text{in } \mathbb{R}^n \\ u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)^{\phi_G} \end{cases} \tag{3.2.2}$$

if $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)^{\phi_G}$.

3.2.1 The Nehari Manifold

As mentioned previously, we approach the existence question for the weighted equation (B) from a varational angle. This involves the approach of Nehari manifolds, which also saw usage by Clapp-Rios [24] for unweighted case $a = b = 0$. Although constrained minimization on Nehari manifolds are now standard, we choose to include the details. This is partially for the sake of completeness, but is also done because literature on the weighted homogeneous Sobolev space $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ is sparse.

Definition 3.2.1. We define $\mathcal{N}^{\phi_G}(\Omega)$ to be the Nehari manifold

$$\mathcal{N}^{\phi_G}(\Omega) := \left\{ u \in \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G} \setminus \{0\} : \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u\|^q_{L^q(\Omega,|x|^{-bq})} \right\}.$$

Subsequently, we set

$$\begin{aligned} c^{\phi_G}(\Omega) &:= \inf_{u\in\mathcal{N}^{\phi_G}(\Omega)} J(u) \\ &= \inf_{u\in\mathcal{N}^{\phi_G}(\Omega)} \left(\frac{1}{p} - \frac{1}{q}\right) \int_\Omega |\nabla u|^p \, |x|^{-ap} \, \mathrm{d}x \geq 0. \end{aligned}$$

By testing a solution u of (3.2.1) against itself, it is easy to see that the Nehari manifold $\mathcal{N}^{\phi_G}(\Omega)$ contains all non-trivial solutions of (3.2.1), should one exist. We also point out that $\mathcal{N}^{\phi_G}(\Omega)$ is non-empty; this is verified in this next lemma which compiles several important properties relating to this Nehari manifold and the minimal energy level $c^{\phi_G}(\Omega)$. We emphasize that the results and arguments in this section are standard (see, for instance, Clapp-Rios [24] and Clapp-Vicente-Benítez [21]), but nonetheless require verification.

Lemma 3.2.2. *There hold the following.*

1. *The Nehari manifold $\mathcal{N}^{\phi_G}(\Omega)$ is non-empty.*

2. *There exists $\mu_0 > 0$ such that*

$$\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \geq \mu_0 \text{ for all } u \in \mathcal{N}^{\phi_G}(\Omega).$$

   Consequently, $c^{\phi_G}(\Omega) > 0$.

3. *We have*

$$c^{\phi_G}(\Omega) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

   where $\Gamma \subseteq C\left([0,1], \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}\right)$ consists of those continuous paths

$$\gamma : [0,1] \to \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$$

beginning at 0 *with non-trivial endpoint of non-positive energy, i.e.*

$$\Gamma := \left\{\gamma \in C\left([0,1], \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}\right) : \gamma(0) = 0,\ \gamma(1) \neq 0,\ J(\gamma(1)) \leq 0\right\}.$$

4. $C_c^\infty(\Omega)^{\phi_G} \cap \mathcal{N}^{\phi_G}(\Omega)$ *is dense in* $\mathcal{N}^{\phi_G}(\Omega)$.

5. *The Nehari manifold* $\mathcal{N}^{\phi_G}(\Omega)$ *is a closed* C^1*-Banach submanifold of* $\mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$, *and a natural constraint for the energy functional* J.

Proof.

1. Certainly, since $C_c^\infty(\Omega)^{\phi_G}$ is non-trivial (it is infinite dimensional) we may choose $u \in C_c^\infty(\Omega)^{\phi_G}$ such that $u \neq 0$. Consider the mapping

$$\ell : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|tu\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|tu\|^q_{L^q(\Omega,|x|^{-bq})}$$

which is clearly continuous. By inspection, it is easy to see that $\ell(t) > 0$ for t near 0 whilst $\ell(t) < 0$ for all $|t|$ sufficiently large. By the Intermediate Value Theorem, it follows that $\ell(t_0 u) = 0$ for some $t_0 \neq 0$ whence $t_0 u \in \mathcal{N}^{\phi_G}(\Omega)$ since, of course, $t_0 u \neq 0$.

2. By the CKN inequality, given $u \in \mathcal{N}^{\phi_G}(\Omega)$, we have

$$\begin{aligned} 0 &= \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|u\|^q_{L^q(\Omega,|x|^{-bq})} \\ &\geq \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - S\,\|u\|^q_{\mathcal{D}_a^{1,p}(\Omega,0)} \\ &= \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}\left(1 - S\,\|u\|^{q-p}_{\mathcal{D}_a^{1,p}(\Omega,0)}\right) \end{aligned}$$

for a suitable constant $S > 0$. Since $q > p$, this is only possible if the $\mathcal{D}_a^{1,p}(\Omega,0)$-norms of $u \in \mathcal{N}^{\phi_G}(\Omega)$ cannot be made arbitrarily close to 0. Thus, there exists $\mu_0 > 0$ such that $\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \geq \mu_0$ for all $u \in \mathcal{N}^{\phi_G}(\Omega)$, as was asserted.

3. First, given $u \in \mathcal{N}^{\phi_G}(\Omega)$ we have that $u \neq 0$ and $\|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u\|^q_{L^q(\Omega,|x|^{-bq})}$

whence, for $t > 1$ sufficiently large,

$$\begin{aligned}J(tu) &= \frac{\|tu\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}}{p} - \frac{\|tu\|^q_{L^q(\Omega,|x|^{-bq})}}{q}\\ &= \left(\frac{t^p}{p} - \frac{t^q}{q}\right)\|u\|_{\mathcal{D}^{1,p}_a(\Omega,0)}\\ &< 0.\end{aligned}$$

Furthermore, by the first derivative test, it is easy to see that $J(tu)$ (with $t > 0$) achieves its maximum at $t = 1$. Let $s_0 > 1$ be such that $J(s_0 u) < 0$ and consider the path

$$\gamma_0 : [0,1] \to \mathcal{D}^{1,p}_a(\Omega,0)^{\phi_G}$$

given by $\gamma_0(t) := ts_0 u$. Then,

$$\begin{cases}\gamma_0(0) = 0,\\ \gamma_0(1) = s_0 u \neq 0,\\ J(\gamma_0(1)) < 0.\end{cases}$$

Thus, $\gamma_0 \in \Gamma$. Observing also that

$$\max_{t\in[0,1]} J(\gamma_0(t)) = \max_{t\in[0,1]} J(ts_0u) = J(u),$$

we infer that

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \max_{t\in[0,1]} J(\gamma_0(t)) = J(u).$$

Since $u \in \mathcal{N}^{\phi_G}(\Omega)$ was arbitrary, it follows that

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \inf_{u\in N^{\phi_G}(\Omega)} J(u) = c^{\phi_G}(\Omega).$$

To establish the reverse inequality, consider the function

$$\kappa : \mathcal{D}^{1,p}_a(\Omega, 0) \to \mathbb{R}, \quad u \mapsto \begin{cases} \dfrac{\|u\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

By appealing to the CKN inequality, this mapping is easily seen to be continuous: indeed, for all $u \in \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G} \setminus \{0\}$ we have

$$|\kappa(u)| = \frac{\|u\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}} \leq S \, \|u\|^{q-p}_{\mathcal{D}^{1,p}_a(\Omega,0)} .$$

Obviously, $\kappa(0) = 0$ while, if $u \in \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$ has non-positive energy (i.e. if $J(u) \leq 0$), then

$$\kappa(u) = \frac{\|u\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)}} \geq \frac{q}{p} > 1.$$

Consequently, for all paths $\gamma \in \Gamma$, we have $\kappa(\gamma(0)) = 0$ and $\kappa(\gamma(1)) > 1$ so that (by the Intermediate Value Theorem) there exists $t_1 \in (0, 1)$ with the property that $\kappa(\gamma(t_1)) = 1$. Then, $\gamma(t_1) \in \mathcal{N}^{\phi_G}(\Omega)$ and

$$\max_{t\in[0,1]} J(\gamma(t)) \geq J(\gamma(t_1)) \geq c^{\phi_G}(\Omega).$$

As this holds for all paths $\gamma \in \Gamma$, we deduce that

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq c^{\phi_G}(\Omega),$$

whence we infer that

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) = c^{\phi_G}(\Omega).$$

4. For this point, let $u \in \mathcal{N}^{\phi_G}(\Omega)$ be given. Then, $u \neq 0$, $u \in \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$, and

$$\|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} = \|u\|^p_{L^q(\Omega,|x|^{-bq})} .$$

Since $C_c^\infty(\Omega)^{\phi_G}$ is dense in $\mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$ by definition, we may select a sequence (u_α) in $C_c^\infty(\Omega)^{\phi_G}$ such that $u_\alpha \to u$ in $\mathcal{D}_a^{1,p}(\Omega,0)$, as $\alpha \to \infty$. Without loss of generality, we may assume that each $u_\alpha \neq 0$. For each index $\alpha \in \mathbb{N}$, we define

$$t_\alpha := \left(\frac{\|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}} \right)^{\frac{1}{q-p}} > 0$$

and set $v_\alpha := t_\alpha u_\alpha$. Then, $v_\alpha \in C_c^\infty(\Omega)^{\phi_G} \subset \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$ for each $\alpha \in \mathbb{N}$ and $v_\alpha \in \mathcal{N}^{\phi_G}(\Omega)$ since

$$\begin{aligned}
\|v_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} &= \|t_\alpha u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \\
&= t_\alpha^p \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \\
&= \left[\frac{\|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}} \right]^{\frac{p}{q-p}} \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \\
&= \left[\frac{\|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}} \right]^{\frac{q}{q-p}} \left[\frac{\|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}}{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}} \right]^{\frac{p-q}{q-p}} \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \\
&= t_\alpha^q \left[\frac{\|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})}}{\|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}} \right]^{\frac{q-p}{q-p}} \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \\
&= t_\alpha^q \|u_\alpha\|^q_{L^q(\Omega,|x|^{-bq})} \\
&= \|v_\alpha\|^q_{L^q(\Omega,|x|^{-bq})} .
\end{aligned}$$

In addition, by continuity of the $\mathcal{D}_a^{1,p}(\Omega,0)$-norm and the CKN inequality, it follows from $u_\alpha \to u$ in $\mathcal{D}_a^{1,p}(\Omega,0)$ that $t_\alpha \to 1$, as $\alpha \to \infty$. Consequently,

$$\begin{aligned}
\|v_\alpha - u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} &\leq \|v_\alpha - u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)} + \|u_\alpha - u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \\
&= \|t_\alpha u_\alpha - u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)} + o(1) \\
&= (t_\alpha - 1) \|u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)} + o(1) \\
&= o(1)
\end{aligned}$$

whence we infer that $v_\alpha \to u$ strongly in $\mathcal{D}_a^{1,p}(\Omega,0)$.

5. This follows by a verbatim extension of the argument used within Clapp-Rios [24, Lemma 2.3]. □

As an important point, we now verify that this minimal energy level $c^{\phi_G}(\Omega)$ is independent of our choice of G-invariant domain Ω. This is crucial, as it ensures that our extracted function (which may a priori be supported on a bounded domain) indeed solves the entire problem posed in (B).

Lemma 3.2.3. *There holds that $c^{\phi_G}(\Omega) = c^{\phi_G}$, where we define $c^{\phi_G} := c^{\phi_G}(\mathbb{R}^n)$.*

Proof. Because $\mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G} \subseteq \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)^{\phi_G}$ after an extension by 0 outside Ω, it is automatic that $c^{\phi_G} \leq c^{\phi_G}(\Omega)$. In order to establish the reverse inequality, let $u \in \mathcal{N}^{\phi_G}(\mathbb{R}^n)$ and select a sequence (u_α) in $C^\infty_c(\mathbb{R}^n)^{\phi_G} \cap \mathcal{N}^{\phi_G}(\mathbb{R}^n)$ converging to u in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ as $\alpha \to \infty$. Since each u_α has compact support and Ω contains the origin, we may select for each index $\alpha \in \mathbb{N}$ some $\lambda_\alpha > 0$ such that the rescaled function

$$v_\alpha(x) := \lambda_\alpha^{-\gamma} u_\alpha\left(\frac{x}{\lambda_\alpha}\right)$$

is compactly supported in Ω. Here, $\gamma > 0$ is the homogeneity exponent, associated to the CKN-inequality, defined in (3.5.8). By this homogeneity property,

$$\|v_\alpha\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} = \|u_\alpha\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} \text{ and } \|u_\alpha\|_{L^q(\mathbb{R}^n,0)} = \|v_\alpha\|_{L^q(\mathbb{R}^n,0)}$$

which implies that $J(v_\alpha) = J(u_\alpha)$ and $v_\alpha \in \mathcal{N}^{\phi_G}(\Omega)$. Finally, observe that

$$c^{\phi_G}(\Omega) \leq J(v_\alpha), \quad \forall \alpha \in \mathbb{N}$$

whence $c^{\phi_G}(\Omega) \leq \lim_{\alpha\to\infty} J(v_\alpha) = \lim_{\alpha\to\infty} J(u_\alpha) = J(u)$. As $u \in \mathcal{N}^{\phi_G}(\mathbb{R}^n)$ is arbitrary, it follows that $c^{\phi_G}(\Omega) \leq c^{\phi_G}$. □

Having developed the necessary groundwork involving this weighted Nehari manifold, the existence of a Palais-Smale sequence now follows from a typical mountainpass argument, which we detail below. We emphasize that the mountain pass theorem only tells us (a priori) that there exists a sequence with is Palais-Smale in the $\mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$ sense.

Corollary 3.2.4 (Existence of (PS)-Sequence at the minimal energy level c^{ϕ_G})**.** *There exists a (PS)-sequence in $\mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$ at energy level c^{ϕ_G}. Namely, there exists a sequence (u_α) in $\mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$ such that*

$$\lim_{\alpha\to\infty} J(u_\alpha) = c^{\phi_G} \quad \textit{and} \quad J'(u_\alpha) \to 0 \ \textit{in} \ \mathcal{D}_a^{-1,p'}(\Omega,0)^{\phi_G}.$$

Proof. By Lemma 3.2.3, it suffices to construct such a (PS)-sequence at the energy level $c^{\phi_G}(\Omega)$. Letting Γ be as in Lemma 3.2.2, notice that given any path $\gamma\in\Gamma$, the composition $t\mapsto J(\gamma(t))$ is a continuous mapping $[0,1]\to\mathbb{R}$ and is therefore bounded. Consequently,

$$c^{\phi_G}(\Omega) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) < \infty.$$

Next, there exists by Lemma 3.2.2 a constant $\mu_0>0$ such that for all $u\in\mathcal{N}^{\phi_G}(\Omega)$ there holds $\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \geq \mu_0$. It follows that

$$\begin{aligned} J(u) = \frac{\|u\|_{\mathcal{D}_a^{1,p}(\Omega,0)}}{p} - \frac{\|u\|_{L^q(\Omega,|x|^{-bq})}}{q} &= \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}\left(\frac{1}{p}-\frac{1}{q}\right) \\ &\geq \mu_0^p\left(\frac{1}{p}-\frac{1}{q}\right) \\ &> 0 \end{aligned}$$

uniformly over $u\in\mathcal{N}^{\phi_G}(\Omega)$. Thus,

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) = c^{\phi_G}(\Omega) \geq \mu_0^p\left(\frac{1}{p}-\frac{1}{q}\right) > 0$$

whilst $J(\gamma(0)) = 0$ and $J(\gamma(1)) \leq 0$ for all $\gamma\in\Gamma$. Consequently, the assertion follows after an application of Theorems 2.8-2.9 from Willem [73]. $\square$

3.3 Palais-Smale Symmetric Criticality

Adhering to the notation used previously, let (G,ϕ_G) be any symmetry imprint satisfying (P1) and $\Omega\subseteq\mathbb{R}^n$ be a G-invariant domain containing the origin. Under these conditions, we have shown in the previous section that we can build a

Palais-Smale sequence at the restricted minimal energy c^{ϕ_G}, within the equivariant subspace $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi_G}$ of $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)$. Since our Struwe-type decompositions in established in Chapter 2 require a Palais-Smale sequence relative to the larger ambient space $\mathcal{D}^{1,p}_a(\Omega,0)$, it is most efficient to show that the sequence we have built is also Palais-Smale in this larger sense, rather than adapt the proofs of Theorems 1.1.1-1.1.2 entirely. This is precisely what the next proposition asserts; the proof of which utilizes a calculation from Clapp-Rios [24, Lemma A.1], together with a straightforward application of Minkowski's inequality for integrals to make the duality connection.

The following proposition, as described above, is a refinement of the principle of symmetric criticality (see Palais [55]) stated for Palais-Smale sequences, rather than mere critical points of the functional.

Proposition 3.3.1. *Suppose $(u_\alpha) \subset \mathcal{D}^{1,p}_a(\Omega,0)^{\phi_G}$ is a Palais-Smale sequence for J in $\mathcal{D}^{1,p}_a(\Omega,0)^{\phi_G}$. That is, as $\alpha \to \infty$, $J(u_\alpha)$ converges to some real number and*

$$J'(u_\alpha) \to 0 \ \textit{in}\ \mathcal{D}^{-1,p'}_a(\Omega,0)^{\phi_G}.$$

Then, (u_α) is a Palais-Smale sequence for J in the non-symmetrized space $\mathcal{D}^{1,p}_a(\Omega,0)$. That is,

$$J'(u_\alpha) \to 0 \ \textit{in}\ \mathcal{D}^{-1,p'}_a(\Omega,0).$$

Moreover, if u solves (3.2.1), then $\langle J'(u),h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\Omega,0)$.

Proof. Let $h \in C^\infty_c(\Omega)$ be an arbitrary test function. Let μ denote the normalized Haar measure on G, so that $\mu(G)=1$, and define

$$\vartheta(x) = \int_G \phi_G(g)h(gx)\,\mathrm{d}\mu.$$

This rule clearly defines a function $\vartheta \in C^\infty_c(\Omega)^{\phi_G}$. Moreover, by Minkowski's inequality for integrals

$$\begin{aligned}\|\vartheta\|_{\mathcal{D}^{1,p}_a(\Omega,0)} &= \left(\int_\Omega \left|\int_G \phi_G(g)g^{-1}\nabla h(gx)\,\mathrm{d}\mu\right|^p |x|^{-ap}\,\mathrm{d}x\right)^{1/p} \\ &\le \int_G \left(\int_\Omega |\phi_G(g)g^{-1}\nabla h(gx)|^p\,|x|^{-ap}\,\mathrm{d}x\right)^{1/p}\mathrm{d}\mu.\end{aligned}$$

Then, simplifying, using that g, g^{-1} are linear isometries, and performing a change of variables, we obtain

$$\begin{aligned}
\|\vartheta\|_{\mathcal{D}_a^{1,p}(\Omega,0)} &\leq \int_G \left(\int_\Omega |\nabla h(gx)|^p \, |gx|^{-ap} \, \mathrm{d}x\right)^{1/p} \mathrm{d}\mu \\
&= \int_G \left(\int_\Omega |\nabla h(y)|^p \, |y|^{-ap} \, \mathrm{d}y\right)^{1/p} \mathrm{d}\mu \\
&= \|h\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \qquad (3.3.1)
\end{aligned}$$

Next, as in Clapp-Rios [24, Lemma A.1], if $u \in \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$ then $u(gx) = \phi_G(g)u(x)$ and $\phi_G(g)\nabla u(x) = g^{-1}\nabla u(gx)$ so

$$\begin{aligned}
&\langle J'(u), \vartheta\rangle \\
&= \int_\Omega |\nabla u(x)|^{p-2} \, \nabla u(x) \cdot \nabla\vartheta(x) \, |x|^{-ap} \, \mathrm{d}x - \int_\Omega |u(x)|^{q-2} \, u(x)\vartheta(x) \, |x|^{-bq} \, \mathrm{d}x \\
&= \int_\Omega \int_G |\nabla u(x)|^{p-2} \, \phi_G(g)\nabla u(x) \cdot (g^{-1}\nabla h(gx)) \, |x|^{-ap} \, \mathrm{d}\mu\mathrm{d}x \\
&\quad - \int_\Omega \int_G |u(x)|^{q-2} \, \phi_G(g)u(x)h(gx) \, |x|^{-bq} \, \mathrm{d}\mu\mathrm{d}x \\
&= \int_\Omega \int_G |\nabla u(gx)|^{p-2} \, (g^{-1}\nabla u(gx)) \cdot (g^{-1}\nabla h(gx)) \, |x|^{-ap} \, \mathrm{d}\mu\mathrm{d}x \\
&\quad - \int_\Omega \int_G |u(x)|^{q-2} \, u(gx)h(gx) \, |x|^{-bq} \, \mathrm{d}\mu\mathrm{d}x.
\end{aligned}$$

Then, using that g^{-1} is a linear isometry and applying Fubini's Theorem, we get

$$\begin{aligned}
&\langle J'(u), \vartheta\rangle \\
&= \int_G \int_\Omega |\nabla u(gx)|^{p-2} \, \nabla u(gx) \cdot \nabla h(gx) \, |gx|^{-ap} \, \mathrm{d}x\mathrm{d}\mu \\
&\quad - \int_G \int_\Omega |u(gx)|^{q-2} \, u(gx)h(gx) \, |gx|^{-bq} \, \mathrm{d}x\mathrm{d}\mu \\
&= \int_G \int_\Omega |\nabla u(y)|^{p-2} \, \nabla u(y) \cdot \nabla h(y) \, |y|^{-ap} \, \mathrm{d}y\mathrm{d}\mu \\
&\quad - \int_G \int_\Omega |u(y)|^{q-2} \, u(y)h(y) \, |y|^{-bq} \, \mathrm{d}y\mathrm{d}\mu \\
&= \langle J'(u), h\rangle .
\end{aligned}$$

In particular,

$$\langle J'(u_\alpha), h\rangle = \langle J'(u_\alpha), \vartheta\rangle .$$

Consequently, if u solves (3.2.1), then $\langle J'(u), h\rangle = 0$ for all $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$. Finally, combining this with (3.3.1) and recalling that $\vartheta \in C^\infty_c(\Omega)^{\phi_G} \subseteq \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$, we see that

$$\begin{aligned}|\langle J'(u_\alpha), h\rangle| = |\langle J'(u_\alpha), \vartheta\rangle| &\leq \|J'(u_\alpha)\|_{\mathcal{D}^{-1,p'}_a(\Omega,0)^{\phi_G}} \|\vartheta\|_{\mathcal{D}^{1,p}_a(\Omega,0)} \\ &\leq \|J'(u_\alpha)\|_{\mathcal{D}^{-1,p'}_a(\Omega,0)^{\phi_G}} \|h\|_{\mathcal{D}^{1,p}_a(\Omega,0)} .\end{aligned}$$

The assertion now follows by a density argument. □

3.4 Symmetric Global Compactness

The goal of this section is to handle the compactness analysis of the carefully chosen Palais-Smale sequence we have built in §3.2, in order to show that this leads to a solution of (B) respecting the given symmetry imprint. Then, appealing to the lifting done by Proposition 3.3.1, we can analyze the asymptotic behaviour of our sequence using a direct application of our previously developed theory from Chapter 2, with the simpler caveat of simply making additional observations along the way.

Again, throughout this section, we denote by (G, ϕ_G) a symmetry imprint for which $(P1) - (P2)$ holds true, and let $\Omega \subset \mathbb{R}^n$ denote a G-invariant domain.

We begin by handling the simpler setting, taking place in the $a < b$ regime.

Proposition 3.4.1. *Assume $a < b$. Suppose there exists a Palais-Smale sequence $(u_\alpha) \subset \mathcal{D}^{1,p}_a(\Omega, 0)^{\phi_G}$ for J such that, as $\alpha \to \infty$,*

$$J(u_\alpha) \to c^{\phi_G}, \quad J'(u_\alpha) \to 0 \text{ in } \mathcal{D}^{-1,p'}_a(\Omega, 0)^{\phi_G}.$$

Then, there exists a non-trivial solution to (3.2.2) which achieves the minimal energy c^{ϕ_G}.

Proof. By an application of Proposition 3.3.1, the sequence (u_α) is Palais-Smale in $\mathcal{D}^{1,p}_a(\Omega, 0)$ with energy level c^{ϕ_G}. Consequently, by following the beginning of the proof for Theorem 1.1.1, there exists $v_0 \in \mathcal{D}_a^{1,p}(\Omega,0)$ such that, after passing to a subsequence, there holds $u_\alpha \rightharpoonup v_0$ in $\mathcal{D}_a^{1,p}(\Omega,0)$,

1. $\nabla u_\alpha \to \nabla v_0$ pointwise a.e. on Ω,
2. $J'(v_0) = 0$ on $\mathcal{D}_a^{1,p}(\Omega,0)$,
3. $J(u_\alpha - v_0) \to c^{\phi_G} - J(v_0)$,
4. $J'(u_\alpha - v_0) \to 0$ in $\mathcal{D}_a^{-1,p'}(\Omega,0)$.

In fact, since $\mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$ is a (weakly) closed subspace of $\mathcal{D}_a^{1,p}(\Omega,0)$, we must have $v_0 \in \mathcal{D}_a^{1,p}(\Omega,0)^{\phi_G}$. Furthermore, it may assumed that $u_\alpha \to v_0$ pointwise a.e. on Ω.

We now distinguish two possible cases. First, if $v_0 \neq 0$, then $v_0 \in \mathcal{N}^{\phi_G}(\Omega)$ so $J(v_0) \geq c^{\phi_G}$ whence $c^{\phi_G} - J(v_0) \leq 0$. However, being a Palais-Smale sequence for J, we see that $(u_\alpha - v_0)$ has non-negative limiting energy, (i.e. by Lemma 2.2.2)

$$\liminf_{\alpha\to\infty} J(u_\alpha - v_0) \geq 0.$$

Hence, we infer that $c^{\phi_G} - J(v_0) \geq 0$ as well yielding $J(v_0) = c^{\phi_G}$. Thus, after an extension by 0, we have $v_0 \in \mathcal{N}^{\phi_G}(\mathbb{R}^n)$ and so (3.2.2) has a non-trivial solution.

It remains to handle the case where $v_0 = 0$. In this case, we continue to follow the proof of Theorem 1.1.1. Then, $v_0 = 0$ and the (PS)-sequence (u_α) has positive energy, we may find sequences (y_α) in $\overline{\Omega}$ and (λ_α) in $(0,\infty)$ such that

$$\delta = \sup_{y\in\overline{\Omega}} \int_{B(y,\lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x = \int_{B(y_\alpha,\lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x$$

where $\delta > 0$ is chosen sufficiently small. Arguing as in Step 4 of [14, Theorem 1], because $a < b$, we can show that (up to a subsequence)

$$\frac{y_\alpha}{\lambda_\alpha} \to x_0$$

for some $x_0 \in \mathbb{R}^n$. We may also assume that $\left| x_0 - \frac{y_\alpha}{\lambda_\alpha} \right| < 1$ for all indices $\alpha \in \mathbb{N}$. Then, setting

$$v_\alpha(x) := \lambda_\alpha^\gamma u_\alpha(\lambda_\alpha x)\,,$$

we obtain a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi_G}$ such that, for all $\alpha \in \mathbb{N}$,

$$\int_{B(x_0,2)} |v_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x \geq \delta.$$

Up to a subsequence, we may suppose (v_α) converges weakly to some $v \in \mathcal{D}^{1,p}_a(\mathbb{R}^n,0)$. By the inequality above, it follows that v is non-trivial by a duality argument (see, for instance, the proof of Step 6 in the proof of Theorem 1.1.1 or Step 3 in Mercuri-Willem [52, Theorem 1.2]). Moreover, arguing as in Step 7 from the proof of Theorem 1.1.1, we obtain $\lambda_\alpha \to 0$ up to a subsequence.

Notice that v is indeed an element of $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi_G}$ since $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi_G}$ is a closed subspace of $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)$ and v is, by definition, the weak limit of a sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi_G}$. Continuing to as in Theorem 1.1.1, we see that $v \in \mathcal{D}^{1,p}_a(\mathbb{R}^n,0)^{\phi_G}$ is a solution to problem (B), and hence to (3.2.2). Furthermore, being that v is non-trivial, $J(v) \geq c^{\phi_G}$ and, defining

$$w_\alpha(z) := u_\alpha(z) - \lambda_\alpha^{-\gamma} v\left(\frac{z}{\lambda_\alpha}\right),$$

we have

1. $J(w_\alpha) \to c^{\phi_G} - J(v)$;

2. $J'(w_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}_a(\Omega,0)$.

As before, $\liminf J(w_\alpha) \geq 0$ and, yet, $c^{\phi_G} - J(v) \leq 0$. It thereby follows that $J(v) = c^{\phi_G}$ and so v is a non-trivial solution to (3.2.2) achieving the energy c^{ϕ_G}. □

We now have everything needed to supply the proof of Theorem 3.1.1, which we provide below.

Proof of Theorem 3.1.1. Observe that, since Corollary 3.2.4 ensures the existence of a Palais-Smale sequence satisfying the requirements of Proposition 3.4.1, we are automatically guaranteed a non-trivial solution to (3.2.2). Then, by applying Proposition 3.3.1, it follows that we have found a solution to (B) which is ϕ_G-equivariant. This concludes the proof. □

We now turn towards the proof of Theorem 3.1.2, which requires an analogue of Proposition 3.4.1 for the case $a = b \neq 0$. As we discussed in detail at the beginning of this chapter (and in Chapter 1), we will require that the symmetry imprint (G, ϕ_G) being considered satisfy the additional orbit condition (P3), in order to ensure that our constructed Palais-Smale sequence concentrate only at the origin, sufficiently rapidly. From a technical view point, we are forcing the concentration to occur exactly as in the $a < b$ regime. In other words, we restrict ourselves to symmetry imprints (G, ϕ_G) which effectively "trap" potential blowup at the origin.

Proposition 3.4.2. *Assume $a = b \neq 0$ and (G, ϕ_G) is such that (P3) holds true. Suppose there exists a Palais-Smale sequence $(u_\alpha) \subset \mathcal{D}_a^{1,p}(\Omega, 0)^{\phi_G}$ for J such that, as $\alpha \to \infty$,*

$$J(u_\alpha) \to c^{\phi_G}, \quad J'(u_\alpha) \to 0 \text{ in } \mathcal{D}_a^{-1,p'}(\Omega, 0)^{\phi_G}.$$

Then, there exists non-trivial solution to (3.2.2) which attains the minimal equivariant energy c^{ϕ_G}.

Proof. As before, Proposition 3.3.1 implies that the given (PS)-sequence is also Palais-Smale when considered within the larger space $\mathcal{D}_a^{1,p}(\Omega, 0)$. Hence, after passing to a subsequence, there exists a weak limit $v_0 \in \mathcal{D}_a^{1,p}(\Omega, 0)$ such that $u_\alpha \rightharpoonup v_0$ in $\mathcal{D}_a^{1,p}(\Omega, 0)$ and pointwise a.e. on Ω, as $\alpha \to \infty$. Moreover,

1. $\nabla u \to \nabla v_0$ pointwise a.e. on Ω and $J'(v_0) = 0$ on $\mathcal{D}_a^{1,p}(\Omega, 0)$,

2. $\phi_G(u_\alpha - v_0) \to c^{\phi_G} - \phi_G(v_0)$,

3. ${\phi_G}'(u_\alpha - v_0) \to 0$ in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$.

Of course, being the weak limit of a sequence in $\mathcal{D}_a^{1,p}(\Omega, 0)^{\phi_G}$, we must have that v_0 is ϕ_G-equivariant, i.e. $v_0 \in \mathcal{D}_a^{1,p}(\Omega, 0)^{\phi_G}$. We now distinguish two possible cases. First, if $v_0 \neq 0$, then we proceed exactly as in the first part of Proposition 3.4.1. If instead $v_0 = 0$, we once again select sequences (y_α) in $\overline{\Omega}$ and (λ_α) in $(0, \infty)$ such that

$$\delta = \sup_{y \in \overline{\Omega}} \int_{B(y, \lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x = \int_{B(y_\alpha, \lambda_\alpha)} |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x$$

where $\delta > 0$ is chosen sufficiently small. Then, by leveraging (P3), we can show that (up to a subsequence)

$$\frac{y_\alpha}{\lambda_\alpha} \to x_0 \tag{3.4.1}$$

for some $x_0 \in \mathbb{R}^n$. Indeed, should this sequence be unbounded, we may pass to a subsequence in such a way that

$$\frac{|y_\alpha|}{\lambda_\alpha} \to \infty, \quad |y_\alpha| > 4\lambda_\alpha.$$

Then, each $y_\alpha \neq 0$ so $y_\alpha / |y_\alpha|$ is a sequence on the unit sphere. Passing to yet another subsequence, we may assume that

$$\frac{y_\alpha}{|y_\alpha|} \to y, \quad \text{as } \alpha \to \infty.$$

Appealing to condition (P3) and borrowing ideas from Clapp-Rios [24], given any $m \in \mathbb{N}$, we may select elements $g_1, \ldots, g_m \in G$ such that $g_i y \neq g_j y$, for all $i \neq j$. Consequently, there exists $\varepsilon > 0$ so that

$$|g_i y - g_j y| \geq 2\varepsilon > 0, \quad \forall i \neq j.$$

Using then that $\frac{y_\alpha}{|y_\alpha|} \to y$, it follows from this last identity that

$$\left| g_i \frac{y_\alpha}{|y_\alpha|} - g_j \frac{y_\alpha}{|y_\alpha|} \right| \geq \varepsilon \tag{3.4.2}$$

for all $\alpha \in \mathbb{N}$ sufficiently large. Hence, for all such α,

$$\frac{|g_i y_\alpha - g_j y_\alpha|}{\lambda_\alpha} \geq \varepsilon \frac{|y_\alpha|}{\lambda_\alpha}. \tag{3.4.3}$$

Because the right hand side tends to $+\infty$, we infer that

$$B(g_i y_\alpha, \lambda_\alpha) \cap B(g_j y_\alpha, \lambda_\alpha) = \varnothing,$$

for all $i \neq j$ and α large. But,

$$\begin{aligned}
\delta &= \int_{B(y_\alpha,\lambda_\alpha)} |u_\alpha|^q\, |x|^{-bq}\, \mathrm{d}x \\
&= \int_{B(g_i y_\alpha,\lambda_\alpha)} \left|u_\alpha(g_i^{-1}z)\right|^q \left|g_i^{-1}z\right|^{-bq} \mathrm{d}z, \qquad\qquad (z := g_i x) \\
&= \int_{B(g_i y_\alpha,\lambda_\alpha)} \left|\phi_G(g_i^{-1})u_\alpha(z)\right|^q |z|^{-bq}\, \mathrm{d}z \\
&= \int_{B(g_i y_\alpha,\lambda_\alpha)} |u_\alpha(z)|^q\, |z|^{-bq}\, \mathrm{d}z
\end{aligned}$$

It thereby follows that, for all α sufficiently large,

$$\int_{\mathbb{R}^n} |u_\alpha|^q\, |x|^{-bq}\, \mathrm{d}x \geq \sum_{i=1}^{m} \int_{B(g_i y_\alpha,\lambda_\alpha)} |u_\alpha(z)|^q\, |z|^{-bq}\, \mathrm{d}z = m\delta,$$

which contradicts the fact that (u_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$. In short: we have proven that y_α/λ_α is bounded and thus, passing to a subsequence, we may assume (3.4.1). From here, we complete the argument by following the second part of the proof from Proposition 3.4.1. □

Equipped with this proposition, we can now prove the general existence theorem for the case $a = b \neq 0$, provided our symmetry imprint satisfies the additional orbit condition (P3).

Proof of Theorem 3.1.2. The proof is exactly the same as for Theorem 3.1.1, but replaces Proposition 3.4.1 with Proposition 3.4.2. □

3.5 A Symmetry Category

Having demonstrated the general capability of producing sign-changing solutions to (B), when provided a symmetry imprint satisfying certain conditions, we have effectively reduced the problem to designing these symmetry imprints. Of course, since the $a < b$ case merely requires that this imprint satisfies (P1)-(P2), our simple example from Example 3.1.1 already provides the existence of nodal solutions in this regime. However, as we discussed previously, this simple approach is insufficient for the interesting (and most difficult) case $a = b \neq 0$, owing to the failure of the infinite orbit condition (P3) for finite groups. In addition, we are also interested in the variety of possible symmetries one can impose on solutions, while retaining the ability to distinguish these solutions using their symmetries. This is precisely the purpose of the $(\alpha, \mathfrak{m})$-symmetries appearing in the statement of Theorem 1.2.1. In fact, we acknowledge once more that the class of $(\alpha, \mathfrak{m})$-symmetries is large enough to bring the existence of new solutions to the unweighted critical p-Laplace problem (C), and to the Yamabe problem (1.1.2).

As stated in the introduction, our groups generalize those obtained by Clapp-Rios [24] and Clapp-Vicente-Benítez [21] by incorporating actions involving non-trivial tuples of complex coordinates of arbitrary length.

Henceforth, we fix a dimension $n \geq 4$ and put $k = k(n) := \lfloor n/2 \rfloor - 1$.

3.5.1 Building Groups of Prescribed Symmetry

For each integer $j \geq 1$, we define an $\mathbb{R}$-linear isometry of $\mathbb{R}^{2(j+1)} \cong \mathbb{C}^{j+1}$ given by

$$\rho_j : \mathbb{C}^{j+1} \to \mathbb{C}^{j+1}, \quad (z_1, \dots, z_{j+1}) \mapsto (-\overline{z_{j+1}}, \overline{z_1}, \dots, \overline{z_j}). \tag{3.5.1}$$

Clearly, ρ_j^ℓ contains a non-trivial permutation of the coordinates $(z_1, \dots, z_{j+1})$ unless $\ell \equiv 0 \mod (j+1)$. Furthermore, when $j+1$ is even, there holds for $z \in \mathbb{C}^{j+1}$,

$$\rho_j^{j+1}(z) = -z \quad \text{and} \quad \rho_j^{2(j+1)}(z) = z$$

On the other hand, it is easily seen that if $j+1$ is odd:

$$\rho_j^{j+1}(z_1, \dots, z_{j+1}) = (-\bar{z}_1, -\bar{z}_2, \dots, -\overline{z_{j+1}}) \quad \text{and} \quad \rho_j^{2(j+1)}(z) = z.$$

Regardless, ρ_j has order $2(j+1)$ for each $j \geq 1$. Additionally, we allow rotations to act upon the coordinates of $\mathbb{C}^{j+1}$ in the synchronous coordinate-wise manner:

$$e^{i\theta}(z_1, \dots, z_{j+1}) := (e^{i\theta} z_1, \dots, e^{i\theta} z_{j+1}),$$

for $\theta \in [0, 2\pi)$. Define Γ_j to be the subgroup of $O(2(j+1))$, acting upon the coordinates of $\mathbb{R}^{2(j+1)} \cong \mathbb{C}^{j+1}$, generated by these rotations and the map ρ_j. We emphasize that, when $j = 1$, this idea comes from Clapp [19] and Clapp-Rios [24].

An easy calculation shows that, for each $\theta \in [0, 2\pi)$,

$$\begin{aligned} e^{i\theta}\rho_j(z_1, \ldots, z_{j+1}) = e^{i\theta}(-\overline{z_{j+1}}, \bar{z}_1, \ldots, \overline{z_j}) &= (-e^{i\theta}\overline{z_{j+1}}, e^{i\theta}\bar{z}_1, \ldots, e^{i\theta}\overline{z_{jj}}) \\ &= (-\overline{e^{-i\theta}z_{j+1}}, \overline{e^{-i\theta}z_1}, \ldots, \overline{e^{-i\theta}z_j}) \\ &= \rho_j e^{i(-\theta)}(z_1, \ldots, z_{j+1}) \\ &= \rho_j e^{i(2\pi-\theta)}(z_1, \ldots, z_{j+1}). \end{aligned}$$

Consequently, rotations permute with ρ_j up to a change in angle in the sense that $e^{i\theta}\rho_j = \rho_j e^{i(2\pi-\theta)}$. In the event that $j+1$ is even, we have $\rho_j^{j+1}(z) = -z$ whence $\rho_j^{j+1} = e^{i\pi}$ is a rotation. However, for all $(j+1)$-odd, no composition of ρ_j can be realized as a rotation (except the identity). It follows from these observations that Γ_j consists precisely of those linear isometries $g \in O(2(j+1))$ uniquely expressible in the form

$$\begin{cases} g = \rho_j^{\varepsilon} e^{i\theta}, & \varepsilon \in \{0, 1, \ldots, j\},\ \theta \in [0, 2\pi) \quad \text{if } j+1 \text{ is even,} \\ g = \rho_j^{\varepsilon} e^{i\theta}, & \varepsilon \in \{0, 1, \ldots, 2j+1\},\ \theta \in [0, 2\pi) \quad \text{if } j+1 \text{ is odd.} \end{cases} \tag{3.5.2}$$

This representation allows us to define a mapping

$$\phi_j : \Gamma_j \to \{\pm 1\}, \quad g \mapsto (-1)^{\varepsilon},$$

where g is expressed uniquely as in (3.5.2); clearly, this is a group homomorphism. Moreover, ϕ_j is continuous being that it is a homomorphism of topological groups that is continuous at the identity. Indeed, consider $\eta_j := (1+i, 1, 0 \ldots, 0) \in \mathbb{C}^{j+1}$ and write $g \in \Gamma_j$ according to (3.5.2). If $(j+1)$ is even and $\varepsilon > 0$, then we have $|g\eta_j - \eta_j| \geq 1$ implying that any g sufficiently close to the identity belongs to the kernel of ϕ_j. Similarly, when $(j+1)$ is odd, we clearly have $|g\eta_j - \eta_j| \geq 1$ unless $\varepsilon = 0$ or $\varepsilon = j+1$. When $\varepsilon = j+1$, direct computation shows that

$$|g\eta_j - \eta_j|^2 = 6 - 4\sin(\theta) + 2\cos(\theta) \geq 6 - 2\sqrt{5}.$$

Thus, again in this case, any $g \in \Gamma_j$ sufficiently close to the identity lies in the kernel of ϕ_j. This demonstrates the continuity of ϕ_j on Γ_j.

Incorporating Pinwheel-type Symmetries

In the following, we borrow ideas from Clapp-Vicente-Benítez [21] (see also Clapp-Faya-Saldanã [23], and Clapp-Saldaña-Soares-Benítez [20]) in order to produce solutions satisfying a "pinwheel-type" symmetry within the weighted case. We remark that whether or not we incorporate such a symmetry is encoded by the value of α from Theorem 1.2.1. More precisely, α will determines the number of "pins" within the symmetry class of our solution.

As in the paper of Clapp-Vicente-Benítez [21], we begin by considering an action on $\mathbb{C}^2 \cong \mathbb{R}^4$. We allow the unit circle to act upon $\mathbb{C}^2$ via asynchronous rotation:

$$e^{i\theta}(z_1, z_2) := (e^{i\theta} z_1, e^{-i\theta} z_2).$$

Next, given a non-negative integer α and $\beta \in \mathbb{N}$, we adopt the transformation

$$\varrho_\alpha^\beta : \mathbb{C}^2 \to \mathbb{C}^2, \quad z \mapsto \cos\left(\frac{\beta\pi}{2^{\alpha+1}}\right) z + \sin\left(\frac{\beta\pi}{2^{\alpha+1}}\right) \rho_1(z).$$

For convenience, we write $\varrho_\alpha := \varrho_\alpha^1$. As illustrated in Clapp-Vicente-Benítez [21], ϱ_α commutes with these rotations and, moreover, ϱ_α has order $2^{\alpha+2}$. In fact, there holds $\varrho_\alpha^\beta \varrho_\alpha^{\beta'} = \varrho_\alpha^{\beta+\beta'}$. If $\alpha = 0$, let

$$\Upsilon_\alpha = \{I\}$$

where I denotes the identity of $O(n)$. Otherwise, let Υ_α denote the copy in $O(n)$ of the $O(4)$-subgroup generated by ϱ_α and these rotations. Next, by the relations above, the mapping

$$\varphi_\alpha : \Upsilon_\alpha \to \{\pm 1\}$$

induced by the rules $\varphi_\alpha(e^{i\theta}) := 1$ and $\varphi_\alpha(\varrho_\alpha) := -1$ is a well defined group homomorphism. If $\alpha = 0$, we let φ_α be the trivial homomorphism. Notice (see, for instance, Clapp-Vicente-Benítez [21]) that Υ_α is compact in $O(n)$ and φ_α is continuous. Furthermore, it is easy to see that the point $(1, 0) \in \mathbb{C}^2$ satisfies the condition $\mathrm{Stab}_{\Upsilon_\alpha}(1, 0) \subseteq \ker \varphi_\alpha$.

3.5.2 Group Construction

Fix now a dimension $n \geq 4$ and a non-negative integer α. Suppose we are given a tuple $\mathfrak{m} := (m_1, \ldots, m_k)$ of non-negative integers and define the following conditions:

Assume henceforth that (1.2.2) is satisfied. For each index $j = 1, \ldots, k$, we denote by $\Gamma_{j,\ell}$, with $\ell = 1, \ldots, m_j$, the copy of Γ_j in $O(n)$ acting upon the coordinates of $\mathbb{R}^n$ indexed by

$$\begin{cases} \displaystyle\sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2(\ell-1)(j+1) + 5 & \text{if } \alpha > 0, \\ \displaystyle\sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2(\ell-1)(j+1) + 1 & \text{if } \alpha = 0, \end{cases}$$

through

$$\begin{cases} \displaystyle\sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2\ell(j+1) + 4 & \text{if } \alpha > 0, \\ \displaystyle\sum_{\iota=1}^{j-1} 2m_\iota(\iota+1) + 2\ell(j+1) & \text{if } \alpha = 0. \end{cases}$$

Since $\Gamma_{j,\ell}$ is an isomorphic copy of Γ_j, we may abuse notation and view ϕ_j as a group homomorphism $\Gamma_{j,\ell} \to \{\pm 1\}$ in the natural way.

According to this scheme, given any non-negative integer α and $\mathfrak{m}$ such that $(\alpha, \mathfrak{m})$ satisfies (1.2.2), we define a subgroup $G_{\alpha,\mathfrak{m}}$ of $O(n)$ by putting

$$G_{\alpha,\mathfrak{m}} := \Upsilon_\alpha \Lambda_{\alpha,\mathfrak{m}} \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}, \tag{3.5.3}$$

where $\Lambda_{\alpha,\mathfrak{m}}$ is a copy of the orthogonal group:

$$\Lambda_{\alpha,\mathfrak{m}} \cong \begin{cases} O\left(n - 4 - 2\sum_{j=1}^{k} m_j(j+1)\right) & \text{if } \alpha > 0 \text{ and (1.2.3) holds}, \\ O\left(n - 2\sum_{j=1}^{k} m_j(j+1)\right) & \text{if } \alpha = 0 \text{ and (1.2.3) holds}, \\ \{I\} & \text{if (1.2.3) fails} \end{cases}$$

acting on the remaining coordinates of $\mathbb{R}^n$, untouched by all of the $\Gamma_{j,\ell}$ and Υ_α. Observe that $G_{\alpha,\mathfrak{m}}$ is indeed a well defined subgroup of $O(n)$ since it is the product of subgroups of $O(n)$, with each subgroup commuting with the others.

It is apparent from the definition in (3.5.3) that each $g \in G_{\alpha,\mathfrak{m}}$ may be uniquely expressed as a product of commutative factors belonging to the respective subgroups appearing in the definition of $G_{\alpha,\mathfrak{m}}$, in the sense that

$$g = \upsilon_\alpha g_0 \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \gamma_{j,\ell}, \quad \upsilon_\alpha \in \Upsilon_\alpha,\ g_0 \in \Lambda_{\alpha,\mathfrak{m}},\ \gamma_{j,\ell} \in \Gamma_{j,\ell}. \tag{3.5.4}$$

Consequently, we define a group homomorphism $\phi_{\alpha,\mathfrak{m}} : G_{\alpha,\mathfrak{m}} \to \{\pm 1\}$ by allowing each ϕ_j to act on the factors of g belonging to the copies of Γ_j, i.e.

$$\phi_{\alpha,\mathfrak{m}}(g) := \varphi_\alpha(\upsilon_\alpha) \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \phi_j(\gamma_{j,\ell}),$$

where g is expressed as in (3.5.4).

Before we proceed with some essential lemmas regarding the properties of the pairs $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,\mathfrak{m}})$ constructed above, let us formally encode some notation which is utilized in the statement Theorem 1.2.1.

Definition 3.5.1. Maintaining the aforedescribed notation, a function $u : \mathbb{R}^n \to \mathbb{R}$ will be said to be $(\alpha, \mathfrak{m})$-symmetric if it is $\phi_{\alpha,\mathfrak{m}}$-equivariant.

Roughly speaking, an $\mathfrak{m}$-type symmetry describes in which ways the solution u, with respect to the remaining coordinates not affected by Υ_α, may be synchronously rotated, reflected, and coordinate swapped in relation to the function's sign. Furthermore, as we shall later demonstrate, these indeed give rise to distinct and incompatible symmetries, in contexts where the assumptions stated in the second part of Theorem 1.2.1 hold.

Remark 3.5.1. It is natural to ask whether one must impose some condition on $\mathfrak{m}$ and $\mathfrak{n}$, as in Theorem 1.2.1-(ii), to distinguish the solutions when $\alpha = \beta$. Without any additional condition, there is no way, in general, to distinguish these solutions using the symmetries alone. For example, the mapping

$$\mathbb{R}^8 \cong \mathbb{C}^4 \to \mathbb{R}, \quad z \mapsto \Im\left(z_1\overline{z_2}z_3\overline{z_4}\right)$$

is an example of a non-trivial function which is both $(0,(2,\mathbf{0}))$-symmetric and $(0,(0,0,1,\mathbf{0}))$-symmetric. Consequently, if we allow for actions on a composite number of complex coordinates, there is no way, in general, to distinguish the resulting symmetry types. In addition, to show that our imposed symmetries are mutually exclusive, we utilize a binary coding scheme together with a gcd-argument requiring that we compare symmetries on a coprime number coordinates. We refer the reader to Lemmas 3.5.2 and 3.5.3, as well as Proposition 3.5.4 for the concrete arguments.

Next, we show that the group and homomorphism pairs constructed above satisfy (P1)-(P2) whenever condition (1.2.2) holds. In addition, we verify that (1.2.3) implies property (P3).

Lemma 3.5.1. *Let $n \geq 4$, $\alpha \geq 0$ an integer, and $\mathfrak{m} := (m_1, \ldots, m_k)$ be a tuple of non-negative integers such that $(\alpha, \mathfrak{m})$ satisfies (1.2.2). Then, the group $G_{\alpha,\mathfrak{m}}$ defined in (3.5.3) is a closed subgroup of $O(n)$, and $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,\mathfrak{m}})$ satisfies conditions (P1)-(P2). Furthermore, if $(\alpha, \mathfrak{m})$ satisfies (1.2.3), then (P3) holds true.*

Proof. We only explicitly treat the case $\alpha > 0$, since a direct adaptation of this argument applies in the special case $\alpha = 0$ (see also the end of this proof for comments on how the proof is adapted in this case).

First we demonstrate that $G_{\alpha,\mathfrak{m}}$ is a closed subgroup of $O(n)$. Indeed, owing to the fact that each ρ_j has finite order, and the set of actions by rotation form a continuous copy of the unit circle in $O(2(j+1))$, it is easy to see that Γ_j is compact in $O(2(j+1))$. It follows that each $\Gamma_{j,\ell}$ is compact in $O(n)$. By Tychonoff's theorem and continuity of the group operation on $O(n)$, we infer that $\Upsilon_\alpha \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}$ is compact in $O(n)$. Then, since $\Lambda_{\alpha,\mathfrak{m}}$ is closed in $O(n)$, it follows from standard theory on topological groups (see Hewitt-Ross [44]) that $G_{\alpha,\mathfrak{m}}$ is closed in $O(n)$, which verifies our first assertion.

By construction, $\phi_{\alpha,\mathfrak{m}}$ is surjective and hence criterion (P2) holds true. In addition, it is easy to see that $\phi_{\alpha,\mathfrak{m}}$ is continuous on $G_{\alpha,\mathfrak{m}}$ since, by our discussions in §2.1-§2.1.1, φ_α and each ϕ_j are continuous. To see that the property (P1) does indeed hold, let $\mathbf{e} = (1, 0) \in \mathbb{C}^2$ and define for each index $j = 1, \ldots, k$ the point $\xi_j = (1 + i, 1, \ldots, 1) \in \mathbb{C}^{j+1}$; consider

$$\xi = (\mathbf{e}, \xi_1, \ldots, \xi_1, \ldots, \xi_k, \ldots, \xi_k, w) \in \mathbb{C}^2 \times \left(\prod_{j=1}^{k} \mathbb{C}^{m_j(j+1)} \right) \times \mathbb{R}^{n-4-2\sum_{j=1}^{k} m_j(j+1)}$$

with $w \in \mathbb{R}^{n-4-2\sum_{j=1}^{k} m_j(j+1)}$ arbitrary. Above, we emphasize that each ξ_j appears exactly m_j-times in the tuple defining ξ. We claim that $\mathrm{Stab}_{G_{\alpha,\mathfrak{m}}}(\xi) \subseteq \ker \phi_{\alpha,\mathfrak{m}}$. Certainly, express any $g \in \mathrm{Stab}_{G_{\alpha,\mathfrak{m}}}(\xi)$ according to (3.5.4) and observe that $g\xi = \xi$ forces $\upsilon_\alpha \mathbf{e} = \mathbf{e}$ and $\gamma_{j,\ell}\xi_j = \xi_j$ for all $j = 1, \ldots, k$ and $\ell = 1, \ldots, m_j$. By our comments at the end of §3.5.1, we have $\upsilon_\alpha \in \ker \varphi_\alpha$. For each admissible pair j, ℓ, write $\gamma_{j,\ell}$ in the form of (3.5.2). That is,

$$\gamma_{j,\ell} = \rho_{j,\ell}^{\varepsilon_{j,\ell}} e^{i\theta_{j,\ell}}$$

with

$$\begin{cases} \varepsilon_{j,\ell} \in \{0, 1, \ldots, j\} & \text{if } j+1 \text{ is even} \\ \varepsilon_{j,\ell} \in \{0, 1, \ldots, 2j+1\} & \text{otherwise.} \end{cases}$$

Here, $\rho_{j,\ell}$ denotes the copy of ρ_j acting upon the coordinates treated by $\Gamma_{j,\ell} \cong \Gamma_j$. Note that, because $\rho_{j,\ell}^{\varepsilon_{j,\ell}}$ shuffles coordinates unless $\varepsilon_{j,\ell} \equiv 0 \mod (j+1)$, we must have either $\varepsilon_{j,\ell} = 0$ or $\varepsilon_{j,\ell} = j + 1$. If $j + 1$ is even, the only possibility is $\varepsilon_{j,\ell} = 0$. If $j + 1$ is odd, then in the event that $\varepsilon_{j,\ell} = j + 1$, the equality $\rho_{j,\ell}^{\varepsilon_{j,\ell}} e^{i\theta_{j,\ell}} \xi_j = \xi_j$ reduces to

$$\left(-e^{-i\theta_{j,\ell}}(1 - i), -e^{-i\theta_{j,\ell}}, -e^{-i\theta_{j,\ell}}, \ldots, -e^{-i\theta_{j,\ell}}\right) = (1 + i, 1, 1, \ldots, 1)$$

which admits no possible solutions. Consequently, we have that each $\varepsilon_{j,\ell} = 0$ whence $g \in \ker \phi_{\alpha,\mathfrak{m}}$ by construction. This verifies (P1).

It remains to establish (P3) when $(\alpha, \mathfrak{m})$ satisfies the additional condition (1.2.3). To this end, notice that for any

$$x = (z, \zeta_{1,1}, \ldots, \zeta_{1,m_1}, \ldots, \zeta_{k,1}, \ldots, \zeta_{k,m_k}, w) \in \mathbb{C}^2 \times \left(\prod_{j=1}^{k} \mathbb{C}^{m_j(j+1)} \right) \times \mathbb{R}^{n-4-2\sum_{j=1}^{k} m_j(j+1)}$$

there holds

$$\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x) = \Upsilon_\alpha z \times \left(\prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}\zeta_{j,\ell} \right) \times \Lambda_{\alpha,\mathfrak{m}} w. \tag{3.5.5}$$

Thus, so long as $x \neq 0$, the cardinality of $\mathrm{Orb}_G(x)$ shall be infinite because condition (1.2.3) ensures that $\Lambda_{\alpha,\mathfrak{m}}$ is infinite in the case $n \neq 4 + 2\sum_{j=1}^{k} m_j(j+1)$. This demonstrates (P3) and concludes the proof.

In the case $\alpha = 0$, we proceed in the same way as above. Indeed, by the same reasoning as before, for $\alpha = 0$ and $\mathfrak{m}$ satisfying (1.2.2), the group $G_{\alpha,\mathfrak{m}}$ given by (3.5.3) defines a closed subgroup of $O(n)$. The corresponding homomorphism is obviously surjective by construction. To verify the stabilizer condition (P1), we use the same argument as in the case $\alpha = 0$, but omitting the inclusion of the point $\mathbf{e} \in \mathbb{C}^2$ and only considering the required copies of $\xi_j \in \mathbb{C}^{j+1}$ within the definition of ξ. If $(\alpha, \mathfrak{m})$ satisfies (1.2.2)-(1.2.3) with $\alpha = 0$, then the orbit of a point

$$x = (\zeta_{1,1}, \ldots, \zeta_{1,m_1}, \ldots, \zeta_{k,1}, \ldots, \zeta_{k,m_k}, w)$$

is given by

$$\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x) = \left(\prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \Gamma_{j,\ell}\zeta_{j,\ell} \right) \times \Lambda_{\alpha,\mathfrak{m}} w$$

and hence (P3) holds true by the same logic as for the case $\alpha > 0$. □

3.5.3 Binary Codes

We now turn towards the task of investigating when the symmetry types $(\alpha, \mathfrak{m})$ satisfying (1.2.2) are incompatible. To achieve this, we leverage a structure which we call a t-code. Although we define this structure in an abstract sense, for the purposes of this monograph, one can think of these as being the tuples encoding which (complex) coordinates one may rotate without altering the value of a function. Essentially, a t-code will be used to describe the coordinates under which a function is invariant, when these coordinates are rotated synchronously.

For any positive integer t, given a binary vector $c := (c_1, \ldots, c_t) \in \mathbb{Z}_2^t$, we shall write $\tilde{c}$ to denote the *cycle* of c given by

$$\tilde{c} := (c_t, c_1, \ldots, c_{t-1}).$$

Definition 3.5.2. A t-code is a non-empty subset $\mathcal{C}$ of the vector space $\mathbb{Z}_2^t$ such that

1. If $c, c' \in C$ with $c \leq c'$ (meaning $c_j \leq c'_j$ for all $j = 1, \ldots, t$) then $c + c' \in \mathcal{C}$.

2. $\tilde{c} \in \mathcal{C}$ for each $c \in \mathcal{C}$.

An element $c \in \mathcal{C}$ is called a *codeword* (of $\mathcal{C}$).

Given an arbitrary integer $0 \leq r \leq t$, it will be convenient to associate it with the codeword

$$v_r := (1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \mathbb{Z}_2^t,$$

whose first r-coordinates are 1 and the remaining $(t - r)$-coordinates are 0. Note that v_0 is simply the zero codeword while v_t is the codeword consisting entirely of ones.

Before presenting the main lemma of this subsection, which constitutes the core of our distinguishing argument, we make a simple observation:

> If $r \leq s \leq t$ are non-negative integers and $v_r, v_s \in \mathcal{C}$ where $\mathcal{C}$ is a t-code, then $v_{s-r} \in \mathcal{C}$

Indeed, if $v_r, v_s \in \mathcal{C}$ then $v_r + v_s \in \mathcal{C}$ and repeatedly cycling $v_r + v_s$ yields $v_{s-r} \in \mathcal{C}$.

Lemma 3.5.2. *Let $r < s \leq t$ be positive integers such that* $\gcd(r, s) = 1$. *Suppose that $\mathcal{C}$ is a t-code containing both v_r and v_s. Then, $\mathcal{C}$ contains the standard basis of $\mathbb{Z}_2^t$.*

Proof. Since $\mathcal{C}$ is closed under cycling, it suffices to show that $v_1 = (1, 0, \ldots, 0) \in \mathcal{C}$. If $r = 1$ there is nothing to show. Otherwise, we write

$$s = k_1 r + r_1$$

for a non-negative integer k_1 and an integer remainder $0 < r_1 < r$. Then, since $v_r, v_s \in \mathcal{C}$ we see that $v_{s-k_1 r} = v_{r_1} \in \mathcal{C}$. If $r_1 = 1$ then we are done. Otherwise, we continue and write

$$r = k_2 r_1 + r_2$$

for a non-negative integer k_2 and an integer remainder $0 < r_2 < r_1$. Since $v_r \in \mathcal{C}$ and $v_{r_1} \in \mathcal{C}$, we also have that $v_{r-k_2 r_1} = v_{r_2} \in \mathcal{C}$. Again, if $r_2 = 1$ then we are done. Otherwise, continuing this process, since $\gcd(r, s) = 1$ the Euclidean algorithm guarantees that, eventually, we have a remainder of 1. So $v_1 \in \mathcal{C}$ and we are done. $\square$

3.5.4 Rotation Codewords

Keeping with the notation presented in this section, let $\alpha \geq 0$ be an integer and $\mathfrak{m}$ a tuple such that $(\alpha, \mathfrak{m})$ satisfies (1.2.2). Let $1 \leq j \leq k$ be such that $m_j > 0$. Fix an index $\ell \in \{1, \ldots, m_j\}$ and denote by $(z_1, z_2, \ldots, z_{j+1})$ the (complex) coordinates the group $\Gamma_{j,\ell} \subseteq G_{\alpha,\mathfrak{m}}$ acts upon. Then, any $x \in \mathbb{R}^n$ may be expressed as

$$x = (w_1, z_1, \ldots, z_{j+1}, w_2),$$

where w_1 and w_2 are real tuples containing, respectively, the coordinates of x before and after $(z_1, \ldots, z_{j+1})$.

Given a binary codeword $c = (c_1, \ldots, c_{j+1}) \in \mathbb{Z}_2^{j+1}$ of length $(j+1)$, we define a rotation operator

$$R_c^{\alpha,\mathfrak{m},j,\ell}(\theta) : \mathbb{R}^n \to \mathbb{R}^n, \quad (w_1, z_1, \ldots, z_{j+1}, w_2) \mapsto (w_1, e^{c_1 i\theta} z_1, \ldots, e^{c_{j+1} i\theta} z_{j+1}, w_2).$$

The key observation is that the set of all $c \in \mathbb{Z}_2^{j+1}$ such that u is invariant under $R_c^{\alpha,\mathfrak{m},j,\ell}(\theta)$ is itself a $(j+1)$-code.

Lemma 3.5.3. *Let $n \geq 4$ and $(\alpha, \mathfrak{m})$ be a pair for which (1.2.2) holds. Suppose $1 \leq j \leq k$ is an index such that $m_j > 0$ and fix $\ell \in \{1, \ldots, m_j\}$. Assume u is $(\alpha, \mathfrak{m})$-symmetric and let $\mathcal{C}$ be the set of all binary codewords $c \in \mathbb{Z}_2^{j+1}$ such that*

$$u(R_c^{\alpha,\mathfrak{m},j,\ell}(\theta)x) = u(x)$$

for all $x \in \mathbb{R}^n$ and $\theta \in [0, 2\pi)$. Then, $\mathcal{C}$ is a $(j+1)$-code.

Proof. For concision, let us write $R_c(\theta) := R_c^{\alpha,\mathfrak{m},j,\ell}(\theta)$ for the duration of the proof. Clearly, $\mathcal{C}$ contains $(0, 0, \ldots, 0)$ by definition. If $c, c' \in \mathcal{C}$ with $c \le c'$ then $c + c' \in \mathcal{C}$ since

$$\begin{aligned} R_c(-\theta)R_{c'}(\theta)(w_1, z_1, \ldots, z_{j+1}, w_2) &= (w_1, e^{(c'_1 - c_1)i\theta} z_1, \ldots, e^{(c'_{j+1} - c_{j+1})i\theta} z_{j+1}, w_2) \\ &= R_{c+c'}(\theta)(w_1, z_1, \ldots, z_{j+1}, w_2) \end{aligned}$$

where $c + c'$ is computed in $\mathbb{Z}_2^{j+1}$. Next, to show $\mathcal{C}$ is closed under cycling, we write

$$\begin{aligned} \rho_j R_c(-\theta)\rho_j^{-1}(w_1, z_1, \ldots, z_{j+1}, w_2) &= \rho_j R_c(-\theta)(w_1, \overline{z_2}, \ldots, \overline{z_{j+1}}, -\overline{z_1}, w_2) \\ &= \rho_j(w_1, e^{-i\theta c_1}\overline{z_2}, \ldots, e^{-i\theta c_j}\overline{z_{j+1}}, -e^{-i\theta c_{j+1}}\overline{z_1}, w_2) \\ &= (w_1, e^{i\theta c_{j+1}} z_1, e^{i\theta c_1} z_2, \ldots, e^{i\theta c_j} z_{j+1}, w_2) \\ &= R_{\tilde{c}}(\theta)x. \end{aligned}$$

In the above, $\rho_j = \rho_{j,\ell}$ is the copy of $\rho_j \in \Gamma_j$ appearing in $\Gamma_{j,\ell}$. Thus, $\tilde{c} \in \mathcal{C}$ and we are done. ◻

Proposition 3.5.4. *Fix a dimension $n \ge 4$. Let $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ be pairs such that each satisfies condition (1.2.2). Assume that $u : \mathbb{R}^n \to \mathbb{R}$ is $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric. If $\alpha = \beta$ and $\mathfrak{m} \neq \mathfrak{n}$, then u vanishes identically if $\mathfrak{m} \lesssim \mathfrak{n}$ or* $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

Proof. We may assume without harm to the proof that $\alpha = \beta = 0$ (otherwise we consider only the coordinates $(x_5, x_6, \ldots, x_n)$ of x). By hypothesis, $\mathfrak{m} \neq \mathfrak{n}$. Thus, there exists a minimal index $j \in \{1, \ldots, k\}$ such that $n_j \neq m_j$. Without loss of generality, suppose we are in the case $m_j < n_j$. We now distinguish two possible settings.

1. If $m_\ell = 0$ for all indices $\ell > j$ then, owing to the fact that $m_j < n_j$, there exists a copy of ρ_j acting on $2(j+1)$-coordinates such that

$$u(\rho_j x) = u(x) \quad \text{and} \quad u(\rho_j x) = -u(x), \quad \forall x \in \mathbb{R}^n,$$

where the first identity arises from the $\phi_{\alpha,\mathfrak{m}}$-equivariance of u and the second comes from the $\phi_{\beta,\mathfrak{n}}$-equivariance. We conclude that $u \equiv 0$.

2. Otherwise, let $\ell > j$ be the minimal index for which $m_\ell > 0$. In this case, there exists a tuple consisting of $2(\ell+1)$-coordinates on which a copy of Γ_ℓ acts, but for which the first $2(j+1)$-coordinates are acted upon by a copy of Γ_j.

   Let now $\mathcal{C}$ denote the set of all binary codewords $c \in \mathbb{Z}_2^{\ell+1}$ such that

$$u(R_c^{\alpha,\mathfrak{m},\ell,1}(\theta)x) = u(x), \quad \forall\theta \in [0, 2\pi)$$

   and all $x \in \mathbb{R}^n$. By Lemma 3.5.3, $\mathcal{C}$ is a $(\ell+1)$-code. From the isomorphic copies of Γ_ℓ and Γ_j, it is automatic that $v_{j+1}, v_{\ell+1} \in \mathcal{C}$. Observe that the condition $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$ enforces $\gcd(\ell+1, j+1) = 1$. An application of Lemma 3.5.2 implies that $\mathcal{C}$ contains the standard basis of $\mathbb{Z}_2^{\ell+1}$, thereby ensuring that u is invariant under rotations in each individual coordinate acted upon by the aforementioned copy of Γ_ℓ.

   By the aforementioned gcd-condition, at least one of $\ell+1$ or $j+1$ must be odd. Thus, we are reduced to two cases.

   (i) If $(\ell+1)$ is odd, then

$$\begin{aligned} u(w_1, z_1, \ldots, z_{\ell+1}, w_2) &= -u(\rho_\ell^{\ell+1}(w_1, z_1, \ldots, z_{\ell+1}, w_2)) \\ &= -u(w_1, -\overline{z_1}, \ldots, -\overline{z_{\ell+1}}, w_2) \end{aligned}$$

   for any $(w_1, z_1, \ldots, z_{\ell+1}, w_2) \in \mathbb{R}^n$. But then, since u is invariant under rotations in each of the (complex) coordinates $z_1, \ldots, z_{\ell+1}$, it follows that

$$u(w_1, -\overline{z_1}, \ldots, -\overline{z_{\ell+1}}, w_2) = u(w_1, z_1, \ldots, z_{\ell+1}, w_2).$$

   Combining the last two equations, we get

$$u(w_1, z_1, \ldots, z_{\ell+1}, w_2) = -u(w_1, z_1, \ldots, z_{\ell+1}, w_2).$$

   We conclude that $u \equiv 0$.

(ii) If $(j+1)$ is odd then, by the same argument,

$$\begin{aligned} u(w_1, z_1, \dots, z_{j+1}, w_2) &= -u(\rho_j^{j+1}(w_1, z_1, \dots, z_{j+1}, w_2)) \\ &= -u(w_1, -\overline{z_1}, \dots, -\overline{z_{j+1}}, w_2) \end{aligned}$$

for any $(w_1, z_1, \dots, z_{\ell+1}, w_2) \in \mathbb{R}^n$. Using that u is rotation invariant in each of the coordinates $z_1, \dots, z_{j+1}, \dots, z_{\ell+1}$, we again find that

$$u(w_1, -\overline{z_1}, \dots, -\overline{z_{j+1}}, w_2) = u(w_1, z_1, \dots, z_{j+1}, w_2).$$

As before, we obtain

$$u(w_1, z_1, \dots, z_{j+1}, w_2) = -u(w_1, z_1, \dots, z_{j+1}, w_2).$$

Ergo, $u \equiv 0$.

In any case, we deduce that $u \equiv 0$ which completes the proof.

□

Proposition 3.5.5. *Let $n \geq 4$ and suppose $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ are two tuples, each satisfying (1.2.2). Let $u : \mathbb{R}^n \to \mathbb{R}$ be $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric. If $\alpha \neq \beta$, then u vanishes identically.*

Proof. Since $\alpha \neq \beta$, we may assume without loss of generality that $\alpha < \beta$. In this case, we must distinguish between $\alpha = 0$ and $\alpha > 0$. In the latter case, we have $\varrho_\alpha = \varrho_\beta^{2^{\beta-\alpha}}$ implying that, for all $x \in \mathbb{R}^n$,

$$u(x) = u(\varrho_\beta^{2^{\beta-\alpha}}(x)) = u(\varrho_\alpha(x)) = -u(x),$$

where the first equality uses the $(\beta, \mathfrak{n})$-symmetry of u and the final equality invokes the $(\alpha, \mathfrak{m})$-symmetry. Consequently, we infer that $u \equiv 0$.

If instead $\alpha = 0$, then observing that $\varrho_\beta^{2^\beta} = \rho_1$ yields a similar contradiction provided $m_1 \neq 0$. Indeed,

$$u(x) = u(\varrho_\beta^{2^\beta}(x)) = u(\rho_1(x)) = -u(x)$$

whence $u \equiv 0$. It remains to treat the setting where $0 = \alpha < \beta$ and $m_1 = 0$. In this case, let $j > 1$ be minimal such that $m_j \neq 0$. Denote by $(z_1, z_2, \dots, z_{j+1})$ the complex coordinates acted upon by the first copy of Γ_j in $G_{\alpha,\mathfrak{m}}$. Then, for any $x = (z_1, \dots, z_{j+1}, w) \in \mathbb{R}^n \cong \mathbb{C}^{j+1} \times \mathbb{R}^{n-2(j+1)}$, there holds

$$u(x) = u(e^{i\theta} z_1, \dots, e^{i\theta} z_{j+1}, w) \tag{3.5.6}$$

for any $\theta \in \mathbb{R}$, by virtue of the $(\alpha, \mathfrak{m})$-symmetry. Owing to the $(\beta, \mathfrak{n})$-symmetry, we also have

$$u(x) = u(e^{i\theta} z_1, e^{-i\theta} z_2, z_3, \dots, z_{j+1}, w)$$

for all angles $\theta \in \mathbb{R}$. By following the argument in Lemma 3.5.3 (i.e. conjugating the above rotations by the copy of ρ_j in $G_{\alpha,\mathfrak{m}}$ acting on the coordinates $(z_1, \dots, z_{j+1})$), we see that

$$\begin{aligned} u(x) &= u(e^{i\theta} z_1, e^{-i\theta} z_2, z_3, \dots, z_j, z_{j+1}, w) && \text{(E.1)} \\ &= u(z_1, e^{i\theta} z_2, e^{-i\theta} z_3, z_4, \dots, z_j, z_{j+1}, w) && \text{(E.2)} \\ &= u(z_1, z_2, e^{i\theta} z_3, e^{-i\theta} z_4, \dots, z_j, z_{j+1}, w) && \text{(E.3)} \\ &\;\;\vdots \\ &= u(z_1, z_2, z_3, z_4, \dots, e^{i\theta} z_j, e^{-i\theta} z_{j+1}, w) && \text{(E.j)} \end{aligned}$$

Now, given θ, we sequentially apply (E.1) with $\theta/(j+1)$, (E.2) with $2\theta/(j+1)$, (E.3) with $3\theta/(j+1)$, and so forth, until we apply (E.j) with $j\theta/(j+1)$. Each such rotation preserves the sign of u, whence

$$u(x) = u(e^{i\theta/(j+1)} z_1, e^{i\theta/(j+1)} z_2, \dots, e^{i\theta/(j+1)} z_j, e^{-ij\theta/(j+1)} z_{j+1}, w).$$

Then, applying (3.5.6) with $-\theta/(j+1)$, we obtain

$$u(x) = u(z_1, z_2, \dots, z_j, e^{-i\theta} z_{j+1}, w).$$

Consequently, u is rotation invariant in the complex coordinate z_{j+1}. Following once more the conjugation argument from Lemma 3.5.3, it follows that u is rotation invariant in each complex coordinate of $(z_1, \dots, z_{j+1})$.

Next, by using that u is $(\beta, \mathfrak{n})$-symmetric with $\beta > 0$, we obtain for every point $x = (z, w) \in \mathbb{R}^n \cong \mathbb{C}^{j+1} \times \mathbb{R}^{n-2(j+1)}$ that

$$\begin{aligned} u(x) = u(\varrho_\beta^{2^\beta}(x)) &= u(-\bar{z}_2, \bar{z}_1, z_3, z_4, \dots, z_j, z_{j+1}, w) \\ &= u(z_2, z_1, z_3, z_4, \dots, z_j, z_{j+1}, w), \end{aligned}$$

where we have used that u is invariant under rotation in each coordinate of z for this last step. Again, by conjugation with respect to ρ_j, the same argument as before (see also Lemma 3.5.3) implies that any two adjacent coordinates of $z = (z_1, \dots, z_{j+1})$ may be swapped while preserving the value of $u(z)$. Iterating this principle yields the identity

$$u(x) = u(z_{j+1}, z_1, z_2, \dots, z_j, w). \tag{3.5.7}$$

On the other hand, the $(\alpha, \mathfrak{m})$-symmetry of u together with the rotational invariance in each coordinate of z ensures that

$$\begin{aligned} -u(x) = u(\rho_j(x)) &= u(-\overline{z_{j+1}}, \bar{z}_1, \dots, \bar{z}_j, w) \\ &= u(z_{j+1}, z_1, \dots, z_j, w). \end{aligned}$$

Combining this last equation with (3.5.7), we conclude that $u(x) = 0$.

In all cases we have shown that $u \equiv 0$, which is what had to be shown. □

Next, we aim to verify that the symmetry incompatibilities we have established in Propositions 3.5.4-3.5.5 also ensure that the solutions we will produce in Theorem 1.2.1 remain distinct even with regards to translations and dilations. This is of important relevance to our problem (B) of interest since, if a function u is a solution to (B), then so is the dilated function $x \mapsto \lambda^\gamma u(\lambda x)$, for $\lambda > 0$, where $\gamma > 0$ is the homogeneity exponent given by

$$\gamma := \frac{n - bq}{q} = \frac{n - p(1 + a)}{p}. \tag{3.5.8}$$

Note that if $a = b = 0$, then translation is allowed in the sense that

$$x \mapsto \lambda^\gamma u(\lambda x + y), \quad y \in \mathbb{R}^n,$$

is also a solution to (B). As a first step in this direction, we show that a non-trivial solution of (B) with $(\alpha, \mathfrak{m})$-symmetry cannot be a translation of a $(\beta, \mathfrak{n})$-symmetric solution, provided $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ satisfy one of the conditions dictated by the second part of Theorem 1.2.1.

Lemma 3.5.6. *Fix a dimension $n \geq 4$. Suppose that $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ are two pairs, each of which satisfies condition (1.2.2). Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be non-trivial solutions to (B) which are $(\alpha, \mathfrak{m})$-symmetric and $(\beta, \mathfrak{n})$-symmetric, respectively. Assume that one of the following conditions hold:*

1. $\alpha \neq \beta$;
2. $\alpha = \beta$ *and* $\mathfrak{m} \neq \mathfrak{n}$, *with* $\mathfrak{m} \lesssim \mathfrak{n}$ *or* $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

Then, u is not a translation of v.

Proof. Suppose for a contradiction that there exists a non-zero $x_0 \in \mathbb{R}^n$ for which there holds $u(x) = v(x - x_0)$ for all $x \in \mathbb{R}^n$. We distinguish two cases.

1. If both $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ satisfy (1.2.3), then the transformation

$$\varsigma : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto -x$$

satisfies the rules

$$u(\varsigma(x)) = u(x) \quad \text{and} \quad v(\varsigma(x)) = v(x)$$

for all $x \in \mathbb{R}^n$. Notice that ς is $\mathbb{R}$-linear. Consequently, for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} u(-x) = u(x) = v(x - x_0) = v(\varsigma(x - x_0)) &= v(-x + x_0) \\ &= u(-x + 2x_0). \end{aligned}$$

As $x \in \mathbb{R}^n$ was arbitrary, we infer that u is periodic and non-trivial; this contradicts u being a non-vanishing solution of (B).

2. Consider next the case where $(\alpha, \mathfrak{m})$ and $(\beta, \mathfrak{n})$ do not both satisfy (1.2.3), and assume without loss of generality that $(\alpha, \mathfrak{m})$ fails this condition. In particular, n must be odd. Then, we consider the transformation

$$\varsigma : \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \dots, x_{n-1}, x_n) \mapsto (-x_1, \dots, -x_{n-1}, x_n).$$

Again, $\varsigma \in G_{\alpha,\mathfrak{m}}$ is an $\mathbb{R}$-linear isometry of $\mathbb{R}^n$. Furthermore, $u(\varsigma x) = u(x)$ for all $x \in \mathbb{R}^n$ because ς is composed of sign-preserving components of $G_{\alpha,\mathfrak{m}}$. In fact, we can say the same for v because

(i) If $(\beta, \mathfrak{n})$ does not satisfy (1.2.3) we get $v(\varsigma x) = v(x)$ by the same reasoning as for u.

(ii) If $(\beta, \mathfrak{n})$ satisfies (1.2.3) there is a non-trivial copy of an orthogonal group acting upon the final coordinates not touched by any sign-changing action from $G_{\beta,\mathfrak{n}}$ whence ς is a sign-preserving transformation in $G_{\beta,\mathfrak{n}}$ with respect to $\phi_{\beta,\mathfrak{n}}$.

Now, $\varsigma^2 = I$ and so

$$\begin{aligned} u(x) = u(\varsigma(x)) = v(\varsigma(x) - x_0) &= v(x - \varsigma(x_0)) \\ &= v(x - \varsigma(x_0) + x_0 - x_0) \\ &= u(x - \varsigma(x_0) + x_0). \end{aligned}$$

Hence, we have periodicity of u provided $\varsigma(x_0) \neq x_0$. Of course, by the same reasoning as earlier, this would be a contradiction. If $\varsigma(x_0) = x_0$, this means that all but the last coordinate of x_0 is 0, at which point we have $\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x_0) = \{x_0\}$ by (1.2.3). But then, $v(x) = u(x + x_0)$ is a translation of u by a $G_{\alpha,\mathfrak{m}}$-invariant point in space. Thus, v is $\phi_{\alpha,\mathfrak{m}}$-equivariant and must be trivial by Propositions 3.5.4-3.5.5.

□

The next corollary verifies the distinctness of our solutions under the previously mentioned translations and dilations.

Corollary 3.5.7. *Fix a dimension* $n \geq 4$. *Suppose that* $(\alpha, \mathfrak{m})$ *and* $(\beta, \mathfrak{n})$ *are two pairs, each of which satisfies condition* (1.2.2). *Let* $u, v : \mathbb{R}^n \to \mathbb{R}$ *be non-trivial solutions to* (B) *which are* $(\alpha, \mathfrak{m})$*-symmetric and* $(\beta, \mathfrak{n})$*-symmetric, respectively. Assume one of the following hold true:*

1. $\alpha \neq \beta$;
2. $\alpha = \beta$ *and* $\mathfrak{m} \neq \mathfrak{n}$, *with* $\mathfrak{m} \lesssim \mathfrak{n}$ *or* $\gcd(\mathfrak{m}, \mathfrak{n}) = 1$.

Then, u *is not a translated rescaling of* v, *in the sense that there does not exist* $\lambda \in \mathbb{R}$ *and* $y \in \mathbb{R}^n$ *such that*

$$u(x) = \lambda^\gamma v(\lambda x + y).$$

Here, $\gamma > 0$ *is the homogeneity exponent given by* (3.5.8).

Proof. By way of contradiction, suppose we can write

$$u(x) = \lambda^\gamma v(\lambda x + y)$$

for some $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Since u is non-trivial, we must have $\lambda \neq 0$. If $y = 0$, then $u(x) = w(x)$ where $w(x) := \lambda^\gamma v(\lambda x)$ is itself $\phi_{\beta,\mathfrak{n}}$-equivariant. Notice also that w is a solution of (B). Consequently, Propositions 3.5.4-3.5.5 imply that u is trivial which contradicts our assumptions. If instead $y \neq 0$, we observe that

$$u(x) = \lambda^\gamma v(\lambda x + y) = \lambda^\gamma v\left(\lambda\left[x + \frac{y}{\lambda}\right]\right) = w\left(x + \frac{y}{\lambda}\right),$$

meaning that u is a translation of a $\phi_{\beta,\mathfrak{n}}$-equivariant solution of (B). Ergo, we are in contradiction with Lemma 3.5.6. Thus, the proof is complete. □

3.6 The Proof of Theorem 1.2.1

We now turn towards the proof of our main existence result for nodal solutions, i.e. we provide the proof of Theorem 1.2.1. As we mentioned in Chapter 1, and in the introductory portion of the current chapter, although we already possess existence results for the problem (1.2.1) in the form of Theorems 3.1.1-3.1.2, the primary pursuit of this chapter was to exhibit nodal solutions to (1.2.1) for an infinite category of mutually exclusive symmetry types. Thus, Theorem 1.2.1 is effectively a combination of these aforementioned general existence results, together with the symmetry discussions from the previous sections in order to distinguish the resulting solutions.

Proof of Theorem 1.2.1. Let $n \geq 4$ and put $k = k(n) := \lfloor \frac{n}{2} \rfloor - 1$. For each $\alpha \geq 0$ and every tuple $\mathfrak{m} = (m_1, \ldots, m_k)$ such that $(\alpha, \mathfrak{m})$ satisfies condition (1.2.2), the construction detailed in §2 yields a symmetry imprint $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,\mathfrak{m}})$ satisfying properties (P1)-(P2) by virtue of Lemma 3.5.1. Then, an application of Theorem 3.1.1 yields a solution $u_{\alpha,m}$ to (B) which is $(\alpha, \mathfrak{m})$-symmetric.

If we are instead in the case $a = b \neq 0$, and the pair $(\alpha, \mathfrak{m})$ satisfies the additional condition (1.2.3), the argument proceeds exactly as in the previous case $a < b$, but instead by appealing to Theorem 3.1.2 since the symmetry imprint $(G_{\alpha,\mathfrak{m}}, \phi_{\alpha,m})$ will satisfy (P3) additionally.

Finally, we treat the case $a = b = 0$. Although condition (P3) will not hold unless (1.2.3) is satisfied, it is easy to see from the definition of $G_{\alpha,\mathfrak{m}}$ and (3.5.5) that the weaker condition

$$\dim\left(\mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x)\right) > 0 \quad \text{or} \quad \mathrm{Orb}_{G_{\alpha,\mathfrak{m}}}(x) = \{x\}$$

is valid for all $x \in \mathbb{R}^n$. Consequently, an application of Clapp-Rios [24, Theorem 3.1] with Ω equal to the unit ball ensures the existence of a non-trivial solution $u_{\alpha,\mathfrak{m}}$ of the problem (C) that is $(\alpha, \mathfrak{m})$-symmetric.

In all cases, being that $\phi_{\alpha,\mathfrak{m}}$ is surjective by construction, it is automatic that each $u_{\alpha,\mathfrak{m}}$ is sign-changing.

The second part of the theorem involves distinguishing the solutions by their symmetry types. Since each solution constructed above is a $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$-function satisfying the Caffarelli-Kohn-Nirenberg problem in $\mathbb{R}^n$, it follows (see Shakerian-Vétois [60]) that u is $C^1(\mathbb{R}^n \setminus \{0\})$, after modification on a set of Lebesgue measure zero. Consequently, by Lemma 3.2.1, each $(\alpha, \mathfrak{m})$-symmetric solution u has a $C(\mathbb{R}^n \setminus \{0\})$ Sobolev representative which, after setting $u(0) = 0$, is a function satisfying the $(\alpha, \mathfrak{m})$-symmetry pointwise everywhere on $\mathbb{R}^n$. Ergo, the second part of the theorem follows by combining Propositions 3.5.4-3.5.5.

$\square$

Chapter 4

Conclusion and Open Questions

There are several natural and interesting followup questions that arise from our discussions in Chapters 1,2 and 3 of this monograph. Beginning perhaps with the global compactness theory established within Chapter 2, and the historical background in Chapter 1, there are several directions involving the Struwe-type decomposition which carry questions of significant interest. First, it would be interesting to investigate, under which conditions, one might be able to sharpen the conclusions of Theorems 1.1.1-1.1.2 by describing the relative formation of the bubbles themselves. To be more precise, one could ask whether bubbles can be segregated during their formation, or if they can cluster at a single concentration point and form a "tower". We comment that this question holds interest in either the $a < b$ or $a = b \neq 0$ regimes, albeit for different reasons. In the $a < b$ regime, because all concentration is automatically enforced at the origin, an investigation into the towering of bubbles is especially relevant. In contrast, when $a = b \neq 0$ in the problem parameters, the fact that a Palais-Smale sequence may split into bubbles solving different categories of problems, makes an investigation of the separation of bubbles particularly interesting. Especially, in the $a = b \neq 0$ regime, it would be interesting to study whether one can construct a sequence exhibiting bubble clusters, at the origin, consisting entirely of unweighted profiles.

Pushing further the questions of sharpening the conclusions of Theorem 1.1.1 and Theorem 1.1.2, we mention also that an investigation of energy bounds for sign changing solutions of the Caffarelli-Kohn-Nirenberg problem (A) in Ω in the spirit of Weth [72] could be useful in establishing the existence of sign-changing solutions to (A), for $\Omega \neq \mathbb{R}^n$. We should comment, however, that the work of Weth [72] is for the unweighted Laplacian, with the proof fundamentally utilizing the conformal invariance of the problem. As this conformal invariance is lost when transitioning to the p-Laplacian, answering such questions would likely require a modified methodology. Closely related to this is the question of comparing the energies of solutions to the weighted problem (B) (for $(a, b) \neq (0, 0)$) when considered relative to the energy of unweighted solutions, i.e. solutions to the problem (C). Note that a comparative result regarding these respective energies could shed some light of when Palais-Smale sequences achieve a splitting into bubbles for *both* the weighted and unweighted limiting problems given by (B) and (C). In fact, identifying suitably general conditions under which a Palais-Smale sequence for the energy functional J in (1.1.1) *cannot* result in unweighted bubbling (in the $a = b \neq 0$ regime) could prove to be useful, in the sense that it may overcome the dimensional constraint $n = 5$ present in the statement of Theorem 1.2.1 (see below as well). One interesting potential path involves using the existence of solutions in $n = 5$ with $a < b$ to build a (PS)-sequence consisting of solutions to a slightly perturbed analogue of the $a = b \neq 0$ case (i.e. with $a = b - \varepsilon$ for some $\varepsilon > 0$ small) in an attempt to produce a (PS)-sequence in the $a = b \neq 0$ regime which inherits the simplified bubble tree from $a < b$, thereby avoiding the algebraic constraints introduced by $n = 5$.

Addressing now interesting follow ups on the existence portion of this thesis, i.e. Chapter 3, we should first discuss the consequences of the dimensional constraint $n \neq 5$ present in the statement of our main existence Theorem 1.2.1 within the balanced case $a = b \neq 0$. The glaring change in this dimension is that we find *no* sign-changing solutions to the problem (1.2.1), rather than the existence of nodal solutions for an infinite number of mutually distinct symmetry classes, as we find in all other cases. We should note here that we do not believe this phenomenon to be a failure of the $(\alpha, \mathfrak{m})$-symmetries themselves. Indeed, when $a = b \neq 0$, we do have a result ensuring the existence of an entire nodal solution to (1.2.1) for a general symmetry imprint satisfying conditions (P1)-(P2)-(P3) (this is Theorem 3.1.2). However, it does not seem trivial to construct a group and homomorphism pair (G, ϕ_G) satisfying all of (P1)-(P2)-(P3) when $n = 5$. What is more, without additional information on the profile decomposition offered by Theorem 1.1.2, condition (P3) is necessary to ensure the forced concentration at the origin – which is essential in verifying that the energy of the Palais-Smale sequence does not escape, nor split, into unweighted bubbles in the limit. As we believe no pair $(G, \phi_G$ satisfying (P1)-(P2)-(P3) can exist in $n = 5$, we suspect that a proof of nodal existence for $n = 5$ and $a = b \neq 0$ would be of interest in it's own right, as it would likely require a new approach differing from the strategy employed within this thesis.

Of interest as well, is how the prescription of symmetries can affect the energies of solutions we constructed. Even whilst remaining within our $(\alpha, \mathfrak{m})$-category of solutions, one could ask how increasing α, or increasing the number of non-zero entries in the class $\mathfrak{m}$, could affect the energies of the solutions we find. That is, can we effectively describe the energies of the solutions using the symmetries we prescribe?

Furthering the question of existence, it would also be interesting to study the implications of our global compactness theory in Chapter 2 in domains other than $\mathbb{R}^n$. For example, it would be interesting to try and obtain a Bahri-Coron-type (i.e. a statement reminiscent of those found in Bahri-Coron [4]) existence result for the problem (A), although this would be most interesting in the situation where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain whose *closure* contains the origin. This class of domains would include, in particular, the case where Ω is an open ball punctured at the origin, as well as domains where the singularity lies on the boundary.

Bibliography

[1] Sérgio Almaraz. The asymptotic behavior of palais–smale sequences on manifolds with boundary. *Pacific Journal of Mathematics*, 269:1 – 17, 2014.

[2] Claudianor O Alves. Existence of positive solutions for a problem with lack of compactness involving the p-laplacian. *Nonlinear Analysis: Theory, Methods & Applications*, 51(7):1187–1206, 2002.

[3] Thierry Aubin. Problèmes isopérimétriques et espaces de sobolev. *Journal of Differential Geometry*, 11:573–598, 1976.

[4] A Bahri and J.M. Coron. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. *Communications on pure and applied mathematics*, 41(3):253–294, 1988.

[5] Giuseppina Barletta, Pasquale Candito, Salvatore A. Marano, and Kanishka Marano, Salvatore A.Perera. Multiplicity results for critical p-laplacian problems. *Journal of Differential Equations*, 265(3):891–905, 2018.

[6] Javier Bracho, Mónica Clapp, and Wacław Marzantowicz. Symmetry breaking solutions of nonlinear elliptic systems. *Toplogical Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center*, 26:189–201, 2005.

[7] Jaeyoung Byeon and Zhi-Qiang Wang. Symmetry breaking of extremal functions for the caffarelli– kohn–nirenberg inequalities. *Communications in Contemporary Mathematics*, 04(03):457–465, 2002.

[8] Luis Caffarelli, Basilis Gidas, and Joel Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth. *Communications on Pure and Applied Mathematics*, 42:271–297, 1989.

[9] Luis Caffarelli, Robert Kohn, and Louis Nirenberg. First order interpolation inequalities with weights. *Compositio Mathematica*, 53(3):259–275, 1984.

[10] Giovanni Catino, Dario D. Monticelli, and Alberto Roncoroni. On the critical p-laplace equation. *Advances in Mathematics*, 433:109331, 2023.

[11] Florin Catrina and Zhi-Qiang Wang. On the caffarelli-kohn-nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Communications on Pure and Applied Mathematics*, 54(2):229–258, 2001.

[12] Edward Chernysh. A global compactness theorem for critical p-laplace equations with weights. Master's thesis, McGill University, Montreal, Quebec, April 2020. Department of Mathematics and Statistics.

[13] Edward Chernysh. Entire nodal solutions with prescribed symmetry to Caffarelli-Kohn-Nirenberg equations. *arXiv*, page 2511.11543, 2025.

[14] Edward Chernysh. A Struwe-type decomposition result for weighted critical p-Laplace equations. *Nonlinear Analysis*, 273:114206, 2026.

[15] Naoki Chiba and Toshio Horiuchi. On radial symmetry and its breaking in the caffarelli-kohn-nirenberg type inequalities for $p = 1$. *Mathematical Journal of Ibaraki University*, 47:49–63, 2015.

[16] Kai Seng Chou and Chiu Wing Chu. On the Best Constant for a Weighted Sobolev-Hardy Inequality. *Journal of the London Mathematical Society*, s2-48(1):137–151, August 1993.

[17] Giulio Ciraolo and Rosario Corso. Symmetry for positive critical points of caffarelli–kohn–nirenberg inequalities. *Nonlinear Analysis*, 216:112683, 2022.

[18] Giulio Ciraolo, Alessio Figalli, and Francesco Maggi. A quantitative analysis of metrics on with almost constant positive scalar curvature, with applications to fast diffusion flows. *International mathematics research notices*, 2018(21):6780–6797, 2018.

[19] Mónica Clapp. Entire nodal solutions to the pure critical exponent problem arising from concentration. *Journal of Differential Equations*, 261(6):3042–3060, 2016.

[20] Mónica Clapp, Alberto Saldaña, Mayra Soares, and Victor A Vicente-Benítez. Optimal pinwheel partitions and pinwheel solutions to a nonlinear Schrödinger system. *Annali di Matematica Pura ed Applicata (1923-)*, 204(4):1443–1468, 2025.

[21] Mónica Clapp and Víctor A Vicente-Benítez. Entire solutions to a quasilinear purely critical competitive system. *Journal of Differential Equations*, 450:113717, 2026.

[22] Mónica Clapp and Tobias Weth. Multiple solutions for the Brezis-Nirenberg problem. *Advances in Differential Equations*, 10(4):463 – 480, 2005.

[23] Mónica Clapp, Jorge Faya, and Alberto Saldaña. Optimal pinwheel partitions for the Yamabe equation. *Nonlinearity*, 37(10):105004, 2024.

[24] Mónica Clapp and Luis Lopez Rios. Entire nodal solutions to the pure critical exponent problem for the p-laplacian. *Journal of Differential Equations*, 265(3):891–905, 2018.

[25] Lucio Damascelli, Susana Merchán, Luigi Montoro, and Berardino Sciunzi. Radial symmetry and applications for a problem involving the $\Delta_p(\cdot)$ operator and critical nonlinearity in $\mathbb{R}^N$. *Advances in Mathematics*, 265:313–335, 2014.

[26] Lucio Damascelli and Mythily Ramaswamy. Symmetry of C^1 solutions of p-laplace equations in $\mathbb{R}^N$. *Advanced Nonlinear Studies*, 1(1):40–64, 2001.

[27] Manuel del Pino, Monica Musso, Frank Pacard, and Angela Pistoia. Large energy entire solutions for the yamabe equation. *Journal of Differential Equations*, 251(9):2568–2597, 2011.

[28] Manuel Del Pino, Monica Musso, Frank Pacard, and Angela Pistoia. Torus action on sn and sign-changing solutions for conformally invariant equations. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 12(1):209–237, 2013.

[29] Bin Deng, Liming Sun, and Juncheng Wei. Sharp quantitative estimates of Struwe's decomposition. *arXiv:2103.15360*, 2021.

[30] Giuseppe Devillanova and Sergio Solimini. Concentration estimates and multiple solutions to elliptic problems at critical growth. *Advances in Differential Equations*, 7(10):1257 – 1280, 2002.

[31] Jean Dolbeault and Maria J. Esteban. Rigidity versus symmetry breaking via nonlinear flows on cylinders and euclidean spaces. *Inventiones mathematicae*, 206:397–440, 2016.

[32] Jean Dolbeault, Maria J. Esteban, Michael Loss, and Gabriella Tarantello. On the symmetry of extremals for the caffarelli-kohn-nirenberg inequalities. *Advanced Nonlinear Studies*, 9(4):713–726, 2009.

[33] Abdallah El Hamidi and Jérôme Vétois. Sharp Sobolev Asymptotics for Critical Anisotropic Equations. *Archive for Rational Mechanics and Analysis*, 192(1):1–36, 2009.

[34] Alberto Farina, Carlo Mercuri, and Michel Willem. A liouville theorem for the p-laplacian and related questions. *Calculus of Variations and Partial Differential Equations*, 58(4):153, 2019.

[35] Veronica Felli and Matthias Schneider. Perturbation results of critical elliptic equations of caffarelli–kohn–nirenberg type. *Journal of Differential Equations*, 191(1):121–142, 2003.

[36] Juan Carlos Fernández, Oscar Palmas, and Jimmy Petean. Supercritical elliptic problems on the round sphere and nodal solutions to the Yamabe problem in projective spaces. *Discrete and Continuous Dynamical Systems*, 40(4):2495–2514.

[37] Juan Carlos Fernández and Jimmy Petean. Low energy nodal solutions to the yamabe equation. *Journal of Differential Equations*, 268(11):6576–6597, 2020.

[38] Alessio Figalli and Federico Glaudo. On the sharp stability of critical points of the Sobolev inequality. *Archive for rational mechanics and analysis*, 237(1):201–258, 2020.

[39] N. Ghoussoub and C. Yuan. Multiple solutions for quasi-linear pdes involving the critical sobolev and hardy exponents. *Transactions of the American Mathematical Society*, 352(12):5703–5743, 2000.

[40] Nassif Ghoussoub and Xiao Song Kang. Hardy-sobolev critical elliptic equations with boundary singularities. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 21, pages 767–793. Elsevier, 2004.

[41] Mohammed Guedda and Laurent Veron. Local and global properties of solutions of quasilinear elliptic equations. *Journal of Differential Equations*, 76(1):159–189, 1988.

[42] Emmanuel Hebey and Frédéric Robert. Coercivity and struwe's compactness for paneitz type operators with constant coefficients. *Calculus of Variations and Partial Differential Equations*, 13:491–517, 2001.

[43] Víctor Hernández-Santamaría and Alberto Saldaña. Existence and convergence of solutions to fractional pure critical exponent problems. *Advanced Nonlinear Studies*, 21(4):827–854, 2021.

[44] E. Hewitt and K.A. Ross. *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*. Grundlehren der mathematischen Wissenschaften. Springer New York, 1994.

[45] Toshio Horiuchi. Best constant in weighted sobolev inequality with weights being powers of distance from the origin. *Journal of Inequalities and Applications [electronic only]*, 1(3):275–292, 1997.

[46] Phuong Le and Diem Hang T. Le. Classification of positive solutions to p-laplace equations with critical hardy–sobolev exponent. *Nonlinear Analysis: Real World Applications*, 74, 2023.

[47] Elliott H. Lieb. Sharp constants in the hardy-littlewood-sobolev and related inequalities. *Annals of Mathematics*, 118(2):349–374, 1983.

[48] Pierre-Louis Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 1. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 1(2):109–145, 1984.

[49] Pierre-Louis Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 2. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 1(4):223–283, 1984.

[50] Saikat Mazumdar. Struwe's decomposition for a polyharmonic operator on a compact riemannian manifold with or without boundary. *Communications on Pure and Applied Analysis*, 16(1):311–330.

[51] Maria Medina and Monica Musso. Doubling nodal solutions to the yamabe equation in rn with maximal rank. *Journal de Mathématiques Pures et Appliquées*, 152:145–188, 2021.

[52] Carlo Mercuri and Michel Willem. A global compactness result for the p-laplacian involving critical nonlinearities. *Discrete and continuous dynamical systems*, 28(2):469–493, 2010.

[53] Morio Obata. The conjectures on conformal transformations of Riemannian manifolds. *Journal of Differential Geometry*, 6(2):247 – 258, 1971.

[54] Qianzhong Ou. On the classification of entire solutions to the critical p-laplace equation. *arXiv preprint*, arXiv:2210.05141, October 2022.

[55] Richard S. Palais. The principle of symmetric criticality. *Communications in Mathematical Physics*, 69(1):19–30, 1979.

[56] Srinivasa Ramanujan. Collected papers of srinivasa ramanujan. 2015.

[57] E. Rodemich. The sobolev inequalities with best possible constants. In *Analysis Seminar*, California Institute of Technology, 1966.

[58] Nicolas Saintier. Asymptotic estimates and blow-up theory for critical equations involving the p-laplacian. *Calculus of Variations and Partial Differential Equations*, 25:299 – 331, 2006.

[59] Berardino Sciunzi. Classification of positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$-solutions to the critical p-laplace equation in $\mathbb{R}^N$. *Advances in Mathematics*, 291:12–23, 2016.

[60] Shaya Shakerian and Jérôme Vétois. Sharp pointwise estimates for weighted critical p-laplace equations. *Nonlinear Analysis*, 206:112236, 2021.

[61] Jacques Simon. Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$. In *Journées d'Analyse Non Linéaire: Proceedings, Besançon, France, June 1977*, pages 205–227. Springer, 2006.

[62] Didier Smets and Michel Willem. Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calculus of Variations and Partial Differential Equations*, 18:57—75, 2003.

[63] Michael Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Mathematische zeitschrift*, 187:511–517, 1984.

[64] Andrzej Szulkin and Shoyeb Waliullah. Sign-changing and symmetry-breaking solutions to singular problems. *Complex Variables and Elliptic Equations*, 57(11):1191–1208, 2012.

[65] Andrzej Szulkin and Michel Willem. On some weakly coercive quasilinear problems with forcing. *Journal d'Analyse Mathématique*, 140:267–281, 2020.

[66] Giorgio Talenti. Best constant in sobolev inequality. *Annali di Matematica pura ed Applicata*, 110:353–372, 1976.

[67] Jérôme Vétois. Multiple solutions for nonlinear elliptic equations on compact riemannian manifolds. *International Journal of Mathematics*, 18(09):1071–1111, 2007.

[68] Jérôme Vétois. A priori estimates and application to the symmetry of solutions for critical p-laplace equations. *Journal of Differential Equations*, 260(1):149–161, 2016.

[69] Jérôme Vétois. A note on the classification of positive solutions to the critical p-laplace equation in $\mathbb{R}^n$. *Advanced Nonlinear Studies*, 24, March 2024.

[70] Juncheng Wei and Yuanze Wu. On the stability of the Caffarelli-Kohn-Nirenberg inequality. *Mathematische Annalen*, 384(3):1509–1546, 2022.

[71] Ding Weiyue. On a conformally invariant elliptic equation on R^n. *Commun. Math. Phys*, 107:331–335, 1986.

[72] Tobias Weth. Energy bounds for entire nodal solutions of autonomous superlinear equations. *Calculus of Variations and Partial Differential Equations*, 27(4):421–437, 2006.

[73] Michel Willem. *Minimax Theorems*, volume 24 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser, Boston, MA, 1996.

[74] Michel Willem. *Functional Analysis: Fundamentals and Applications*. Cornerstones. Birkhäuser, New York, NY, 2013.

[75] Chunyan Zhang, Pei Ma, and Tiantian Zheng. Entire sign-changing solutions to the fractional p-laplacian equation involving critical nonlinearity. *Nonlinear Analysis*, 235:113346, 2023.